Game Theory I

Decisions with conflict
What is game theory?

• Mathematical models of conflicts of interest involving:
  – Outcomes (and utility preferences thereon)
  – Actions (single or multiple)
  – Observations of state of game (complete, partial, or probabilistic-beliefs)
  – Model of other actors (especially important if other players actions are not observable at the time of decision.

• Players are modeled as attempting to maximize their utility of outcomes by selecting an action strategy
  – Strategy: an action sequence plan contingent on observations made at each step of the game
  – Mixed strategy: a probabilistic mixture of determinate strategies.
What can Game Theory model (potentially)?

• Economic behavior
  – Contracts, markets, bargaining, arbitration…

• Politics
  – Voter behavior, Coalition formation, War initiation,…

• Sociology
  – Group decision making
  – Social values: fairness, altruism, reciprocity, truthfulness
  – Social strategies: Competition, Cooperation Trust
  – Mate selection
  – Social dominance (Battle of the sexes with unequal payoffs)
Game Formulations-Game rules

• Game rules should specify
  – Game tree-- all possible states and moves articulated
  – Partition of tree by players
  – Probability distributions over all chance moves
  – Characterization of each player’s Information set
  – Assignment of a set of outcomes to each terminal node in the tree.

• Example: GOPS or Goofspiel
  – Two players. deck of cards is divided into suits, Player A gets Hearts, B gets diamonds. Spades are shuffled and uncovered one by one. Goal-- Get max value in spades. On each play, A and B vie for the uncovered spade by putting down a card from their hand. Max value of the card wins the spade.
Goofspiel with hidden 1st player card

A’s move

e.g. Actual play

What should B’s move be?
Game Tree for 3-card Goofspiel, A’s move hidden

Player A’s outcome

3 spades is revealed

Player A’s information Set, move 1

S= Random Shuffle & Deal of Spades=> 6 possible initial game states
Viewing initial 2 reduces game state to 2 possibilities

Actual Spade sequence After shuffle

B’s info, move 1

S

D L W D
2 3 2 3

W L D W
1 3 1 3

W W W W
1 2 1 2

123

231

312

321

132
Games in Normal Form

• Enumerate all possible strategies
  – Each strategy is a planned sequence of moves, contingent on each information state.
  – Example:
    • A strategy: play Spade +1 (with 1 played for 3)
    • B strategy: match 1st spade, then play larger 2 remaining cards if A plays 3 first. Otherwise, play the smaller.

<table>
<thead>
<tr>
<th>Deck</th>
<th>Player A</th>
<th>Player B</th>
<th>Cards Won by A</th>
<th>Outcome for A</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>231</td>
<td>123</td>
<td>1,2</td>
<td>Draw</td>
</tr>
<tr>
<td>132</td>
<td>213</td>
<td>123</td>
<td>1,draw on 2</td>
<td>Lose</td>
</tr>
<tr>
<td>213</td>
<td>321</td>
<td>231</td>
<td>2, draw on 3</td>
<td>Win</td>
</tr>
<tr>
<td>231</td>
<td>312</td>
<td>231</td>
<td>2,1</td>
<td>Draw</td>
</tr>
<tr>
<td>312</td>
<td>123</td>
<td>312</td>
<td>1,2</td>
<td>Draw</td>
</tr>
<tr>
<td>321</td>
<td>132</td>
<td>3122</td>
<td>2, draw on 1</td>
<td>Lose</td>
</tr>
</tbody>
</table>
Definition: normal-form or strategic-form representation

The *normal-form (or strategic-form)* **representation** of a game $G$ specifies:

- A finite set of players $\{1, 2, \ldots, n\}$,
- players’ strategy spaces $S_1, S_2, \ldots, S_n$ and
- their payoff functions $u_1, u_2, \ldots, u_n$
  where $u_i : S_1 \times S_2 \times \ldots \times S_n \rightarrow \mathbb{R}$. 
Games in Normal Form (2 player)

- Make a table with all pairs of event contingent strategies, and place in the cell the values of the outcomes for both players.

<table>
<thead>
<tr>
<th>A’s</th>
<th>B’s</th>
<th>S₁</th>
<th>⋯</th>
<th>Sₙ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>S₁</td>
<td>⋯</td>
<td>S₁</td>
</tr>
<tr>
<td>S₁</td>
<td>U₁(₁₁), U₂(₁₁)</td>
<td>⋯</td>
<td>U₁(₁₁), U₂(₁₁)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sₘ</td>
<td>U₁(₁ₘ), U₂(₁ₘ)</td>
<td>U₁(₁ₘ), U₂(₁ₘ)</td>
</tr>
</tbody>
</table>

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Normal-form representation: 2-player game

- Bi-matrix representation
  - 2 players: Player 1 and Player 2
  - Each player has a finite number of strategies
- Example:
  \[ S_1 = \{ s_{11}, s_{12}, s_{13} \} \quad S_2 = \{ s_{21}, s_{22} \} \]
- (Outcomes of pairs of strategies assumed known)

<table>
<thead>
<tr>
<th>Player 1</th>
<th>( s_1 )</th>
<th>( s_{12} )</th>
<th>( s_{13} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1 )</td>
<td>( u_1(s_{11}, s_{21}) ), ( u_2(s_{11}, s_{21}) )</td>
<td>( u_1(s_{12}, s_{21}) ), ( u_2(s_{12}, s_{21}) )</td>
<td>( u_1(s_{13}, s_{21}) ), ( u_2(s_{13}, s_{21}) )</td>
</tr>
<tr>
<td>Player 2</td>
<td>( S_2 )</td>
<td>( s_{21} )</td>
<td>( s_{22} )</td>
</tr>
<tr>
<td>----------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>( u_1(s_{11}, s_{21}) ), ( u_2(s_{11}, s_{21}) )</td>
<td>( u_1(s_{12}, s_{21}) ), ( u_2(s_{12}, s_{21}) )</td>
<td>( u_1(s_{13}, s_{21}) ), ( u_2(s_{13}, s_{21}) )</td>
<td></td>
</tr>
</tbody>
</table>
Classic Example: Prisoners’ Dilemma

- Two suspects held in separate cells are charged with a major crime. However, there is not enough evidence.
- Both suspects are told the following policy:
  - If neither confesses then both will be convicted of a minor offense and sentenced to one month in jail.
  - If both confess then both will be sentenced to jail for six months.
  - If one confesses but the other does not, then the confessor will be released but the other will be sentenced to jail for nine months.

<table>
<thead>
<tr>
<th>Prisoner 1</th>
<th>Mum</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mum</td>
<td>-1, -1</td>
<td>-9, 0</td>
</tr>
<tr>
<td>Confess</td>
<td>0, -9</td>
<td>-6, -6</td>
</tr>
</tbody>
</table>

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Example: The battle of the sexes

- At the separate workplaces, Chris and Pat must choose to attend either an opera or a prize fight in the evening.
- Both Chris and Pat know the following:
  - Both would like to spend the evening together.
  - But Chris prefers the opera.
  - Pat prefers the prize fight.

```
<table>
<thead>
<tr>
<th></th>
<th>Opera</th>
<th>Prize Fight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chris</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Pat</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>
```
Example: Matching pennies

- Each of the two players has a penny.
- Two players must *simultaneously* choose whether to show the Head or the Tail.
- Both players know the following rules:
  - If two pennies match (both heads or both tails) then player 2 wins player 1’s penny.
  - Otherwise, player 1 wins player 2’s penny.

```
<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td></td>
</tr>
<tr>
<td>-1 , 1</td>
<td>1 , -1</td>
</tr>
<tr>
<td>Tail</td>
<td></td>
</tr>
<tr>
<td>1 , -1</td>
<td>-1 , 1</td>
</tr>
</tbody>
</table>
```
Static (or simultaneous-move) games of complete information

A static (or simultaneous-move) game consists of:

- A set of players (at least two players)
- For each player, a set of strategies/actions
- Payoffs received by each player for the combinations of the strategies, or for each player, preferences over the combinations of the strategies

\[
\begin{align*}
\text{A} & \text{ set of players (at least two players)} \\
\text{F} & \text{or each player, a set of strategies/actions} \\
\text{P} & \text{ayoffs received by each player for the combinations of the strategies, or for each player, preferences over the combinations of the strategies}
\end{align*}
\]
Static (or simultaneous-move) games of complete information

- **Simultaneous-move**
  - Each player chooses his/her strategy without knowledge of others’ choices.

- **Complete information**
  - Each player’s strategies and payoff function are common knowledge among all the players.

- **Assumptions on the players**
  - **Rationality**
    - Players aim to maximize their payoffs
    - Players are perfect calculators
  - Each player knows that other players are rational
Static (or simultaneous-move) games of complete information

• The players cooperate?
  ➢ No. Only noncooperative games

• The timing
  ➢ Each player $i$ chooses his/her strategy $s_i$ without knowledge of others’ choices.

➢ Then each player $i$ receives his/her payoff
  $u_i(s_1, s_2, \ldots, s_n)$.

➢ The game ends.
Classic example: Prisoners’ Dilemma: normal-form representation

- Set of players: \{Prisoner 1, Prisoner 2\}
- Sets of strategies: \(S_1 = S_2 = \{\text{Mum, Confess}\}\)
- Payoff functions:
  
  \[
  \begin{align*}
  u_1(M, M) &= -1, & u_1(M, C) &= -9, & u_1(C, M) &= 0, & u_1(C, C) &= -6; \\
  u_2(M, M) &= -1, & u_2(M, C) &= 0, & u_2(C, M) &= -9, & u_2(C, C) &= -6
  \end{align*}
  \]

<table>
<thead>
<tr>
<th></th>
<th>Mum</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mum</td>
<td>-1</td>
<td>-9</td>
</tr>
<tr>
<td>Confess</td>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td></td>
<td>-9</td>
<td>-6</td>
</tr>
</tbody>
</table>
Example: The battle of the sexes

- Normal (or strategic) form representation:
  - Set of players: \{Chris, Pat\} (=\{Player 1, Player 2\})
  - Sets of strategies: \(S_1 = S_2 = \{\text{Opera, Prize Fight}\}\)
  - Payoff functions:
    \[
    \begin{align*}
    u_1(O, O) &= 2, & u_1(O, F) &= 0, & u_1(F, O) &= 0, & u_1(F, F) &= 1; \\
    u_2(O, O) &= 1, & u_2(O, F) &= 0, & u_2(F, O) &= 0, & u_2(F, F) &= 2
    \end{align*}
    \]
Example: Matching pennies

- Normal (or strategic) form representation:
  - Set of players: \{Player 1, Player 2\}
  - Sets of strategies: \( S_1 = S_2 = \{\text{Head, Tail}\} \)
  - Payoff functions:
    \[
    u_1(H, H) = -1, \quad u_1(H, T) = 1, \quad u_1(T, H) = 1, \quad u_1(H, T) = -1; \\
    u_2(H, H) = 1, \quad u_2(H, T) = -1, \quad u_2(T, H) = -1, \quad u_2(T, T) = 1
    \]

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
<tr>
<td>Tail</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

Player 1   
<table>
<thead>
<tr>
<th></th>
<th>Head</th>
<th>Tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
<tr>
<td>Tail</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>
## Games for eliciting social preferences

### Table 1: Seven experimental games useful for measuring social preferences

<table>
<thead>
<tr>
<th>Game</th>
<th>Definition of the Game</th>
<th>Real life Example</th>
<th>Predictions with rational and selfish players</th>
<th>Experimental regularities, References</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prisioners’ dilemma Game</td>
<td>Two players, each of whom can either cooperate or defect. Payoff’s are as follows:</td>
<td>Production of negative externality (pollution, loud noise), exchange</td>
<td>Defect</td>
<td>Reciprocate expected cooperation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Cooperate: H, H</td>
<td>without binding contracts, status competition.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Defect: S, T</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>H-L, T-H, L-S</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Public Goods Game</td>
<td>n players simultaneously decide about their contribution g_i (0&lt;s_i&lt;1) where y is</td>
<td>Team compensation, cooperative production in simple societies, overuse of</td>
<td>Each player contributes nothing, i.e. g_i = 0</td>
<td>Players contribute 50% of y in the</td>
<td>Reciprocate expected cooperation</td>
</tr>
<tr>
<td></td>
<td>players’ endowment; each player i earns π_i = y - g_i + mG where G is the sum of all</td>
<td>common resources (e.g., water, fishing grounds)</td>
<td></td>
<td>one-shot game. Contributions unravel</td>
<td></td>
</tr>
<tr>
<td></td>
<td>contributions and m=1 to n</td>
<td></td>
<td></td>
<td>over time. Majority chooses g_i=0 in</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>final period. Communication strongly</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>increases cooperation. Individual</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>punishment opportunities greatly</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>increase contributions.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Ledyard (1995)**.</td>
<td></td>
</tr>
<tr>
<td>Ultimatum Game</td>
<td>Division of a fixed sum of money S between a proposer and a Responder. Proposer</td>
<td>Monopoly pricing of a perishable good; “15 minute” settlement offers</td>
<td>Offer x = e; where e is the smallest money</td>
<td>Responders punish unfair offers;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>offers x. If Responder rejects x both earn zero, if x is accepted the proposer</td>
<td>before a time deadline</td>
<td>unit. Any x&gt;0 is accepted.</td>
<td>negative reciprocity</td>
<td></td>
</tr>
<tr>
<td></td>
<td>earns S - x and the responder earns x.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dictator Game</td>
<td>Like the ultimatum game but the responder cannot reject, i.e., the proposer</td>
<td>Charitable sharing of a windfall gain (lottery winners giving anonymously to</td>
<td>No sharing, i.e., x = 0</td>
<td>On-average “Proposers” allocate</td>
<td>Pure altruism</td>
</tr>
<tr>
<td></td>
<td>dictates (S-x, x).</td>
<td>strangers)</td>
<td></td>
<td>x=25. Strong variations across</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>experiments and across individuals</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Kahaneman et al (1986)*, Camerer</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(2003)**</td>
<td></td>
</tr>
</tbody>
</table>
## More Games

<table>
<thead>
<tr>
<th>Trust Game</th>
<th>Investor has endowment $S$ and makes a transfer $y$ between 0 and $S$ to the Trustee. Trustee receives $3y$ and can send back any $x$ between 0 and $3y$. Investor earns $S - y + x$. Trustee earns $3y - x$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sequential exchange without binding contracts (buying from sellers on Ebay)</td>
<td>Trustee repays nothing: $x = 0$. Investor invests nothing: $y = 0$.</td>
</tr>
<tr>
<td>On average $y = .5S$ and trustees repay slightly less than $.5S$. $x$ is increasing in $y$.</td>
<td>Trustees show positive reciprocity.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Gift Exchange Game</th>
<th>“Employer” offers a wage $w$ to the “worker” and announces a desired effort level $e$. If worker rejects $(w, e)$ both earn nothing. If worker accepts, he can choose any $e$ between 1 and 10. Then employer earns $10s - w$ and worker earn $w - c(e)$. $c(e)$ is the effort cost which is strictly increasing in $e$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Noncontractibility or nonenforceability of the performance (effort, quality of goods) of workers or sellers.</td>
<td>Worker chooses $e = 1$. Employer pays the minimum wage.</td>
</tr>
<tr>
<td>Effort increases with the wage $w$. Employers pay wages that are far above the minimum. Workers accept offers with low wages but respond with $e = 1$. In contrast to the ultimatum game competition among workers (i.e., Responders) has no impact on wage offers.</td>
<td>Workers reciprocate generous wage offers. Employers appeal to workers’ reciprocity by offering generous wages.</td>
</tr>
<tr>
<td>Fehr et al (1993)*</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Third Party Punishment Game</th>
<th>A and B play a dictator game. C observes how much of amount $S$ is allocated to B. C can punish A but the punishment is also costly for C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Social disapproval of unacceptable treatment of others (scolding neighbors).</td>
<td>A allocates nothing to B. C never punishes A.</td>
</tr>
<tr>
<td>Punishment of A is the higher the less A allocates to B.</td>
<td>C sanctions violation of a sharing norm.</td>
</tr>
<tr>
<td>Fehr and Fischbacher (2001a)*</td>
<td></td>
</tr>
</tbody>
</table>

Note: ** denotes survey papers. * denotes papers that introduced the respective games.
Core Concepts we Need from Game Theory

- **Strategy**
- **Mixed strategy**
- **Information set**
- Dominance
- Nash Equilibrium
- Subgame Perfection
- Types of Players (Bayesian games)
“Understanding” a Game

Fundamental assumption of game theory:

Get Rid of the Strictly Dominated strategies. They Won’t Happen.

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>-1, -1</td>
<td>-9, 0</td>
</tr>
<tr>
<td>D</td>
<td>0, -9</td>
<td>-6, -6</td>
</tr>
</tbody>
</table>

In some cases (e.g. prisoner’s dilemma) this means, if players are “rational” we can predict the outcome of the game.
Definition: strictly dominated strategy

In the normal-form game \( \{S_1, S_2, \ldots, S_n, u_1, u_2, \ldots, u_n\} \), let \( s_i', s_i'' \in S_i \) be feasible strategies for player \( i \). Strategy \( s_i' \) is **strictly dominated** by strategy \( s_i'' \) if

\[
\forall s_1 \in S_1, s_2 \in S_2, \ldots, s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, \ldots, s_n \in S_n
\]

\[
u_i(s_1, s_2, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n) < u_i(s_1, s_2, \ldots, s_{i-1}, s_i'', s_{i+1}, \ldots, s_n)
\]

for all \( s_1 \in S_1, s_2 \in S_2, \ldots, s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, \ldots, s_n \in S_n \).

### Prisoner's Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Mum</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mum</td>
<td>-1</td>
<td>-9</td>
</tr>
<tr>
<td>Confess</td>
<td>0</td>
<td>-6</td>
</tr>
</tbody>
</table>

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Definition: weakly dominated strategy

In the normal-form game \( \{S_1, S_2, ..., S_n, u_1, u_2, ..., u_n\} \), let \( s_i', s_i'' \in S_i \) be feasible strategies for player \( i \). Strategy \( s_i' \) is \textbf{weakly dominated} by strategy \( s_i'' \) if

\[
u_i(s_1, s_2, ... s_{i-1}, s_i', s_{i+1}, ..., s_n) \leq \text{(but not always =)} \ u_i(s_1, s_2, ... s_{i-1}, s_i'', s_{i+1}, ..., s_n)
\]

for all \( s_1 \in S_1, s_2 \in S_2, ..., s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, ..., s_n \in S_n \).

\( s_i'' \) is at least as good as \( s_i' \)

regardless of other players’ choices

\[\begin{array}{c|cc}
\text{Player 1} & \text{L} & \text{R} \\
\hline
\text{U} & 1, 1 & 2, 0 \\
\text{B} & 0, 2 & 2, 2 \\
\end{array}\]
Strictly and weakly dominated strategy

• A rational player never chooses a strictly dominated strategy (that it perceives). Hence, any strictly dominated strategy can be eliminated.

• A rational player may choose a weakly dominated strategy.
“Understanding” a Game

Fundamental assumption of game theory:

Get Rid of the Strictly Dominated strategies. They Won’t Happen.

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
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</thead>
<tbody>
<tr>
<td>C</td>
<td>-1, -1</td>
<td>-9, 0</td>
</tr>
<tr>
<td>D</td>
<td>0, -9</td>
<td>-6, -6</td>
</tr>
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</table>

In some cases (e.g. prisoner’s dilemma) this means, if players are “rational” we can predict the outcome of the game.

Several of these slides from Andrew Moore’s tutorials http://www.cs.cmu.edu/~awm/tutorials
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“Understanding” a Game

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\[
\begin{array}{cc}
C & D \\
C & -1, -1 & -9, 0 \\
D & 0, -9 & -6, -6 \\
\end{array}
\]

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Fundamental assumption of game theory:

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\[
\begin{array}{c|cc}
& C & D \\
\hline
C & -1, -1 & -9, 0 \\
D & 0, -9 & -6, -6 \\
\end{array}
\]

In some cases (e.g. prisoner’s dilemma) this means, if players are “rational” we can predict the outcome of the game.
### Strict Domination Removal Example

#### Player B

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>3,1</td>
<td>4,1</td>
<td>5,9</td>
<td>2,6</td>
</tr>
<tr>
<td>II</td>
<td>5,3</td>
<td>5,8</td>
<td>9,7</td>
<td>9,3</td>
</tr>
<tr>
<td>III</td>
<td>2,3</td>
<td>8,4</td>
<td>6,2</td>
<td>6,3</td>
</tr>
<tr>
<td>IV</td>
<td>3,8</td>
<td>3,1</td>
<td>2,3</td>
<td>4,5</td>
</tr>
</tbody>
</table>

So is strict domination the best tool for predicting what will transpire in a game?
Strict Domination doesn’t capture the whole picture

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0,4</td>
<td>4,0</td>
<td>5,3</td>
</tr>
<tr>
<td>II</td>
<td>4,0</td>
<td>0,4</td>
<td>5,3</td>
</tr>
<tr>
<td>III</td>
<td>3,5</td>
<td>3,5</td>
<td>6,6</td>
</tr>
</tbody>
</table>

What strict domination eliminations can we do?

What would you predict the players of this game would do?
Nash Equilibria

\[ S_1^* \in S_1, S_2^* \in S_2, \ldots S_n^* \in S_n \]

are a NASH EQUILIBRIUM iff

\[ \forall i \quad S_i^* = \arg \max_{s_i} u_i \left( S_1^*, S_2^*, \ldots S_{i-1}^*, S_i, S_{i+1}^* \ldots S_n^* \right) \]

\[
\begin{array}{c|c|c}
  & I_b & II_b & III_b \\
 I_a & 0 & 4 & 5 \\
 II_a & 4 & 0 & 3 \\
 III_a & 3 & 5 & 6 \\
\end{array}
\]

\[ u_1(III_a, III_b) = \max \begin{bmatrix} u_1(I_a, III_b) \\ u_1(II_a, III_b) \\ u_1(III_a, III_b) \end{bmatrix} \]

AND \[ u_2(III_a, III_b) = \max \begin{bmatrix} u_2(III_a, I_b) \\ u_2(III_a, II_b) \\ u_2(III_a, III_b) \end{bmatrix} \]

(\text{III}_a, \text{III}_b) is a N.E. because

Several of these slides from Andrew Moore’s tutorials http://www.cs.cmu.edu/~awm/tutorials
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• If \((S_1^*, S_2^*)\) is an N.E. then player 1 won’t want to change their play given player 2 is doing \(S_2^*\)

• If \((S_1^*, S_2^*)\) is an N.E. then player 2 won’t want to change their play given player 1 is doing \(S_1^*\)

Find the NEs:

\[
\begin{array}{ccc}
-1 & -1 & -9 & 0 \\
0 & -9 & -6 & -6 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 4 & 4 & 0 \\
4 & 0 & 0 & 4 \\
3 & 5 & 3 & 5 \\
\end{array}
\]

• Is there always at least one NE?
• Can there be more than one NE?
Example with no NEs among the pure strategies:

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0, 1</td>
<td>1, 0</td>
</tr>
<tr>
<td>$S_2$</td>
<td>1, 0</td>
<td>0, 1</td>
</tr>
</tbody>
</table>
2-player mixed strategy Nash Equilibrium

The pair of mixed strategies \((M_A, M_B)\) are a Nash Equilibrium iff

- \(M_A\) is player A’s best mixed strategy response to \(M_B\)
- AND
- \(M_B\) is player B’s best mixed strategy response to \(M_A\)
Fundamental Theorems

• In the n-player pure strategy game $G=\{S_1, S_2, \ldots, S_n; u_1, u_2, \ldots, u_n\}$, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(S_1^*, S_2^*, \ldots, S_n^*)$ then these strategies are the unique NE of the game.

• Any NE will survive iterated elimination of strictly dominated strategies.

• [Nash, 1950] If $n$ is finite and $S_i$ is finite $\forall i$, then there exists at least one NE (possibly involving mixed strategies).
Back to the Battle

Two Nash Equilibria

Payoffs
\[ M_1 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \]

Mixed strategies
\[ p_1 = \begin{bmatrix} \alpha \\ 1 - \alpha \end{bmatrix}, p_2 = \begin{bmatrix} \beta \\ 1 - \beta \end{bmatrix} \]

Values
\[ u_1 = p_1^T M_1 p_2 \]
\[ u_2 = p_1^T M_2 p_2 \]
What is Fair?

1/4, 1/4

Battle of the Sexes!
Bargaining- Agreeing to Eliminate strategy pairs

Fair-Flip a coin and Agree to let coin-flip be binding.

Requires a coordinated decision- Chris and Pat have to talk to achieve this.
Nash Equilibria Being Useful

THE TRAGEDY OF THE Commons

- You graze goats on the commons to eventually fatten up and sell
- The more goats you graze the less well fed they are
- And so the less money you get when you sell them
Commons Facts

Price = \sqrt{36 - G}

Selling Price per Goat

<table>
<thead>
<tr>
<th>G</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.00</td>
</tr>
<tr>
<td>10</td>
<td>5.00</td>
</tr>
<tr>
<td>20</td>
<td>4.00</td>
</tr>
<tr>
<td>30</td>
<td>3.00</td>
</tr>
<tr>
<td>36</td>
<td>0.00</td>
</tr>
</tbody>
</table>

How many goats would one rational farmer choose to graze?

What would the farmer earn?

What about a group of *n* individual farmers?

Answering this...

...is good practice for answering this

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$n$ farmers

$i$’th farmer has an infinite space of strategies

$$g_i \in [0, 36]$$

An outcome of

$$(g_1, g_2, g_3 \cdots, g_n)$$

will pay how much to the $i$’th farmer?

$$g_i \times \sqrt{36 - \sum_{j=1}^{n} g_j}$$
Let's assume a pure Nash Equilibrium exists.

Call it 

\((g_1^*, g_2^*, \ldots g_n^*)\)

What can we say about \(g_1^*\) ?

\[g_i^* = \arg \max_{g_i} \left[ \text{Payoff to farmer } i, \text{ assuming} \right. \]
\[\left. \text{the other players play} \right] \left( g_1^*, g_2^*, \ldots g_{i-1}^*, g_{i+1}^*, \ldots g_n^* \right) \]

For notational convenience,

write \(G_{-i}^* = \sum_{j \neq i} g_j^*\)

then

\[g_i^* = \arg \max_{g_i} \left[ g_i \sqrt{36 - g_i - G_{-i}^*} \right] \]
Let's Assume a pure Nash Equilibrium exists.

Call it 
\[(g_1^*, g_2^*, \ldots, g_n^*)\]

What can we say about \(g_i^*\)? We can write
\[
g_i^* = \arg \max_{g_i} \text{Payoff of } g_i^* \text{ at } (g_1^*, g_2^*, \ldots, g_n^*)
\]

For Notational Convenience write \(G_i^* = \sum_{j \neq i} g_j^*\)

THEN
\[
g_i^* = \arg \max_{g_i} \left[ g_i \sqrt{36 - g_i^* - G_i^*} \right]
\]

\(g_i^*\) must satisfy
\[
\frac{\partial}{\partial g_i^*} g_i^* \sqrt{36 - g_i^* - G_i^*} = 0
\]

Therefore
\[
36 - G_i^* - \frac{3}{2} g_i^* \sqrt{36 - g_i^* - G_i^*} = 0
\]
We have $n$ linear equations in $n$ unknowns

\[ g_1^* = 24 - \frac{2}{3}(g_2^* + g_3^* + \cdots g_n^*) \]
\[ g_2^* = 24 - \frac{2}{3}(g_1^* + g_3^* + \cdots g_n^*) \]
\[ g_3^* = 24 - \frac{2}{3}(g_1^* + g_2^* + g_4^* \cdots g_n^*) \]
\[ \vdots \]
\[ g_n^* = 24 - \frac{2}{3}(g_1^* + g_2^* + \cdots g_{n-1}^*) \]

Clearly all the $g_i^*$'s are the same (Proof by “it’s bloody obvious”)

Write \( g^* = g_1^* = \cdots g_n^* \)

Solution to \( g^* = 24 - \frac{2}{3}(n-1)g^* \) is:

\[
g^* = \frac{72}{2n+1}
\]
Consequences

At the Nash Equilibrium a rational farmer grazes
\[ \frac{72}{2n+1} \] goats.

How many goats in general will be grazed? Trivial algebra gives:
\[ \frac{36}{2n+1} \] goats total being grazed

\[ \text{as } n \to \text{infinity, } \#\text{goats} \to 36 \]

How much profit per farmer?
\[ \frac{432}{(2n+1)^{3/2}} \]

1.26¢ if 24 farmers

How much if the farmers could all cooperate?
\[ \frac{24\sqrt{12}}{n} = \frac{83.1}{n} \]

3.46¢ if 24 farmers

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The Tragedy

The farmers act “rationally” and earn 1.26 cents each. But if they’d all just got together and decided “one goat each” they’d have got 3.46 cents each.

Is there a bug in Game Theory? in the Farmers? in Nash?

Would you recommend the farmers hire a police force?
Less Tragic with Repeated Plays?

• Does the Tragedy of the Commons matter to us when we’re analyzing human behavior?
• Maybe repeated play means we can learn to cooperate??
Repeated Games with Implausible Threats

Takeo and Randy are stuck in an elevator
Takeo has a $1000 bill
Randy has a stick of dynamite
Randy says “Give me $1000 or I’ll blow us both up.”

What should Takeo do??????

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Using the formalism of Repeated Games With Implausible Threats, Takeo should **Not** give the money to Randy

Takeo **Assumes Randy is Rational**

At this node, Randy will choose the left branch

**Repeated Games**

Suppose you have a game which you are going to play a finite number of times.

What should you do?
2-Step Prisoner’s Dilemma

GAME 1

<table>
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<tr>
<td>D</td>
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<td>-6, -6</td>
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</table>

GAME 2

(Played with knowledge of outcome of GAME 1)

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<tbody>
<tr>
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</tr>
<tr>
<td>D</td>
<td>0, -9</td>
<td>-6, -6</td>
</tr>
</tbody>
</table>

Idea 1

Player A has four pure strategies:
- C then C
- C then D
- D then C
- D then D

Is Idea 1 correct?

Ditto for B
Important Theoretical Result:

Assuming Implausible Threats, if the game $G$ has a unique N.E. $(s_1^*, \ldots, s_n^*)$ then the new game of repeating $G$ $T$ times, and adding payouts, has a unique N.E. of repeatedly choosing the original N.E. $(s_1^*, \ldots, s_n^*)$ in every game.

If you’re about to play prisoner’s dilemma 20 times, you should defect 20 times.

DRAT 😞
Example: mutually assured destruction

- Two superpowers, 1 and 2, have engaged in a provocative incident. The timing is as follows.
- The game starts with superpower 1’s choice either ignore the incident (I), resulting in the payoffs (0, 0), or to escalate the situation (E).
- Following escalation by superpower 1, superpower 2 can back down (B), causing it to lose face and result in the payoffs (1, -1), or it can choose to proceed to an atomic confrontation situation (A). Upon this choice, the two superpowers play the following simultaneous move game.
- They can either retreat (R) or choose to doomsday (D) in which the world is destroyed. If both choose to retreat then they suffer a small loss and payoffs are (-0.5, -0.5). If either chooses doomsday then the world is destroyed and payoffs are (-K, -K), where K is very large number.
Example: mutually assured destruction

$$\begin{align*}
I & : (1, -1) \\
E & : (1, 0) \\
B & : (0, 0) \\
A & : (2, -1) \\
R & : (-0.5, -0.5) \\
D & : (-K, -K)
\end{align*}$$
Subgame

- A subgame of a dynamic game tree
  - begins at a singleton information set (an information set contains a single node), and
  - includes all the nodes and edges following the singleton information set, and
  - does not cut any information set; that is, if a node of an information set belongs to this subgame then all the nodes of the information set also belong to the subgame.
Subgame: illustration

The diagram illustrates a subgame with various nodes and branches, showing the payoffs at each node.
Subgame-perfect Nash equilibrium

• A Nash equilibrium of a dynamic game is subgame-perfect if the strategies of the Nash equilibrium constitute or induce a Nash equilibrium in every subgame of the game.

• Subgame-perfect Nash equilibrium is a Nash equilibrium.
Find subgame perfect Nash equilibria: backward induction

- Starting with those smallest subgames
- Then move backward until the root is reached

One subgame-perfect Nash equilibrium (IR, AR)
Find subgame perfect Nash equilibria: backward induction

- Starting with those smallest subgames
- Then move backward until the root is reached

Another subgame-perfect Nash equilibrium (\(ED, BD\))
Bayesian Games

You are Player A in the following game. What should you do?

```
   S₁  S₂
S₁  3   -2
S₂  0   6
```

**Question:** When does this situation arise?
Recipe for Nash-Equilibrium-Based Analysis of Such Games

• Assume you’ve been given a problem where the i’th player chooses a real number \( x_i \).
• Guess the existence of a Nash equilibrium \( (x_1^*, x_2^* \ldots x_n^*) \).
• Note that, \( \forall i \),

\[
x^*_i = \arg \max_{x_i} \left\{ \begin{array}{l}
\text{Payoff to player } i \text{ if player } i \\
\text{plays } "x_i" \text{ and the } j\text{'th player}
\end{array} \right.
\begin{array}{l}
\text{plays } x^*_j \text{ for } j \neq i
\end{array}
\]

• Hack the algebra, often using “at } x^*_i \text{ we have}
Hockey lovers get 2 units for watching hockey, and 1 unit for watching football.

Football lovers get 2 units for watching football, and 1 unit for watching hockey.

Pat’s a hockey lover.

Pat thinks Chris is probably a hockey lover also, but Pat is not sure.

<table>
<thead>
<tr>
<th></th>
<th>Chris</th>
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<tbody>
<tr>
<td><strong>H</strong></td>
<td>H</td>
</tr>
<tr>
<td><strong>F</strong></td>
<td>F</td>
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</tbody>
</table>

<table>
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<th>H</th>
<th>F</th>
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<tbody>
<tr>
<td><strong>H</strong></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td><strong>F</strong></td>
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</table>

With 2/3 chance

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1/3 chance

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In a Bayesian Game each player is given a type. All players know their own types but only a prob. dist. for their opponent’s types.

An $n$-player Bayesian Game has:

- A set of action spaces $A_1 \ldots A_n$.
- A set of type spaces $T_1 \ldots T_n$.
- A set of beliefs $P_1 \ldots P_n$.
- A set of payoff functions $u_1 \ldots u_n$.

$P_{-i}(t_{-i}|t_i)$ is the prob dist of the types for the other players, given player $i$ has type $i$.

$u_i(a_1, a_2 \ldots a_n, t_i)$ is the payout to player $i$ if player $j$ chooses action $a_j$ (with $a_j \in A_j$) (forall $j=1,2,\ldots,n$) and if player $i$ has type $t_i \in T_i$. 
Bayesian Games: Who Knows What?

We assume that all players enter knowing the full information about the $A_i$’s, $T_i$’s, $P_i$’s and $u_i$’s. The $i$’th player knows $t_i$, but not $t_1 \ t_2 \ t_3 \ \cdots \ t_{i-1} \ t_{i+1} \ \cdots \ t_n$.

All players know that all other players know the above.

And they know that they know that they know, ad infinitum.

**Definition:** A strategy $S_i(t_i)$ in a Bayesian Game is a mapping from $T_i \ ? \ A_i$ : a specification of what action would be taken for each type.
Example

\[ A_1 = \{H,F\} \quad A_2 = \{H,F\} \]
\[ T_1 = \{H-love,F-love\} \quad T_2 = \{H-love, F-love\} \]
\[ P_1 \left( t_2 = H-love \mid t_1 = H-love \right) = \frac{2}{3} \]
\[ P_1 \left( t_2 = F-love \mid t_1 = H-love \right) = \frac{1}{3} \]
\[ P_1 \left( t_2 = H-love \mid t_1 = F-love \right) = \frac{2}{3} \]
\[ P_1 \left( t_2 = F-love \mid t_1 = H-love \right) = \frac{1}{3} \]
\[ P_2 \left( t_1 = H-love \mid t_2 = H-love \right) = 1 \]
\[ P_2 \left( t_1 = F-love \mid t_2 = H-love \right) = 0 \]
\[ P_2 \left( t_1 = H-love \mid t_2 = F-love \right) = 1 \]
\[ P_2 \left( t_1 = F-love \mid t_2 = H-love \right) = 0 \]

\[ u_1 \left( H,H,H-love \right) = 2 \quad u_2 \left( H,H,H-love \right) = 2 \]
\[ u_1 \left( H,H,F-love \right) = 1 \quad u_2 \left( H,H,F-love \right) = 1 \]
\[ u_1 \left( H,F,H-love \right) = 0 \quad u_2 \left( H,F,H-love \right) = 0 \]
\[ u_1 \left( H,F,F-love \right) = 0 \quad u_2 \left( H,F,F-love \right) = 0 \]
\[ u_1 \left( F,H,H-love \right) = 0 \quad u_2 \left( F,H,H-love \right) = 0 \]
\[ u_1 \left( F,H,F-love \right) = 0 \quad u_2 \left( F,H,F-love \right) = 0 \]
\[ u_1 \left( F,F,H-love \right) = 1 \quad u_2 \left( F,F,H-love \right) = 1 \]
\[ u_1 \left( F,F,F-love \right) = 2 \quad u_2 \left( F,F,F-love \right) = 2 \]
Bayesian Nash Equilibrium

The set of strategies \((s_1^*, s_2^*, \ldots, s_n^*)\) are a

Pure Strategy Bayesian Nash Equilibrium

iff for each player \(i\), and for each possible type of \(i: t_i \in T_i\)

\[
s_i^*(t_i) = \arg \max_a \sum_{t_{-i} \in T_{-i}} u_i(s_1^*(t_1), \ldots, s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), \ldots, s_n^*(t_n)) \times P_i(t_{-i} | t_i)
\]

i.e. no player, in any of their types, wants to change their strategy
NEGOTIATION: A Bayesian Game

Two players: S, (seller) and B, (buyer)

$T_s = [0,1]$ the seller’s type is a real number between 0 and 1 specifying the value (in dollars) to them of the object they are selling

$T_b = [0,1]$ the buyer’s type is also a real number. The value to the buyer.

Assume that at the start

$V_s \in T_s$ is chosen uniformly at random

$V_b \in T_b$ is chosen uniformly at random
The “Double Auction” Negotiation

S writes down a price for the item \((g_s)\)
B simultaneously writes down a price \((g_b)\)
Prices are revealed
If \(g_s = g_b\) no trade occurs, both players have payoff 0
If \(g_s = g_b\) then buyer pays the midpoint price \[\frac{(g_s + g_b)}{2}\] and receives the item
Payoff to S: \[\frac{1}{2}(g_s + g_b) - V_s\]
Payoff to B: \[V_b - \frac{1}{2}(g_s + g_b)\]
Negotiation in Bayesian Game Notation

\[ T_s = [0, 1] \text{ write } V_s \in T_s \]
\[ T_b = [0, 1] \text{ write } V_b \in T_b \]
\[ P_s(V_b | V_s) = P_s(V_b) = \text{ uniform distribution on } [0, 1] \]
\[ P_b(V_s | V_b) = P_b(V_s) = \text{ uniform distribution on } [0, 1] \]
\[ A_s = [0, 1] \text{ write } g_s \in A_s \]
\[ A_b = [0, 1] \text{ write } g_b \in A_b \]

\[ u_s(P_s, P_b, V_s) = \text{ What?} \]

\[ u_b(P_s, P_b, V_b) = \text{ What?} \]
Double Negotiation: When does trade occur?

...when

\[ g_b^*(V_b) = \frac{1}{12} + \frac{2}{3} V_b > \frac{1}{4} + \frac{2}{3} V_s = g_s^*(V_s) \]

i.e. when \( V_b > V_s + \frac{1}{4} \)

\[ \text{Prob(Trade Happens)} = \frac{1}{2} \times (\frac{3}{4})^2 = \frac{9}{32} \]
What You Should Know

Strict dominance
Nash Equilibria
Continuous games like Tragedy of the Commons
Rough, vague, appreciation of threats
Bayesian Game formulation
What You Shouldn’t Know

• How many goats your lecturer has on his property
• What strategy Mephistopheles uses in his negotiations
• What strategy this University employs when setting tuition
• How to square a circle using only compass and straight edge
• How many of your friends and colleagues are active Santa informants, and how critical they’ve been of your obvious failings