Game Theory I

Decisions with conflict

What is game theory?

- Mathematical models of conflicts of interest involving:
  - Outcomes (and utility preferences thereon)
  - Actions (single or multiple)
  - Observations of state of game (complete, partial, or probabilistic-beliefs)
  - Model of other actors (especially important if other players actions are not observable at the time of decision.
- Players are modeled as attempting to maximize their utility of outcomes by selecting an action strategy
  - Strategy: an action sequence plan contingent on observations made at each step of the game
  - Mixed strategy: a probabilistic mixture of determinate strategies.

What can Game Theory model (potentially)?

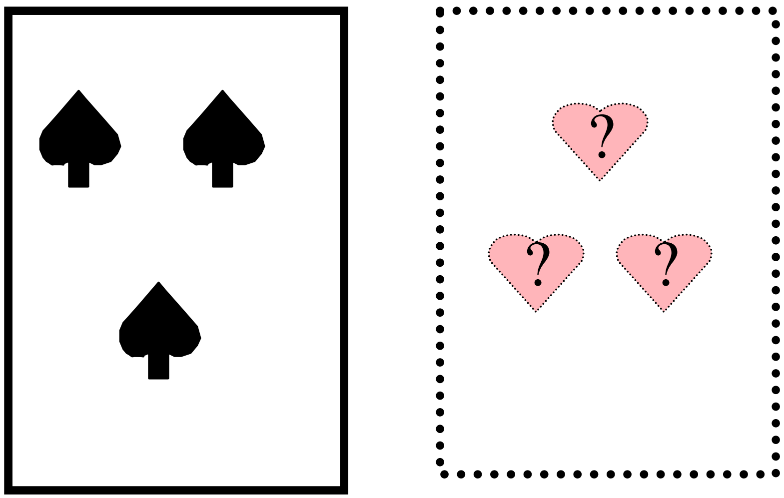
- Economic behavior
  - Contracts, markets, bargaining, arbitration…
- Politics
  - Voter behavior, Coalition formation, War initiation,…
- Sociology
  - Group decision making
  - Social values: fairness, altruism, reciprocity,truthfulness
  - Social strategies: Competition, Cooperation Trust
  - Mate selection
  - Social dominance (Battle of the sexes with unequal payoffs)

Game Formulations-Game rules

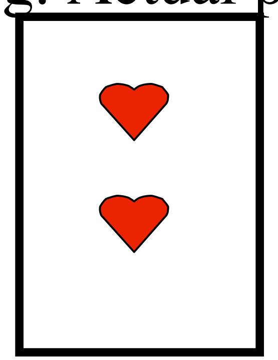
- Game rules should specify
  - Game tree-- all possible states and moves articulated
  - Partition of tree by players
  - Probability distributions over all chance moves
  - Characterization of each player's Information set
  - Assignment of a set of outcomes to each terminal node in the tree.
- Example: GOPS or Goofspiel
  - Two players. deck of cards is divided into suits, Player A gets Hearts, B gets diamonds. Spades are shuffled and uncovered one by one. Goal-- Get max value in spades. On each play, A and B vie for the uncovered spade by putting down a card from their hand. Max value of the card wins the spade.

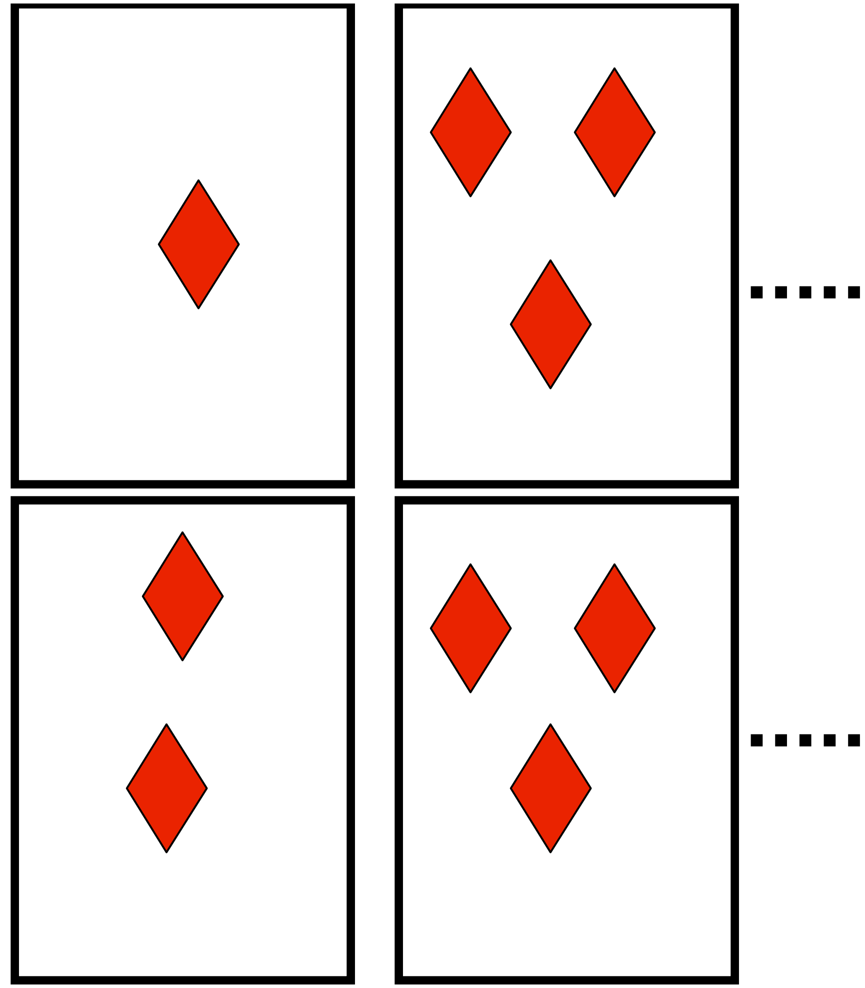
Goofspiel with hidden 1st player card

A’s move



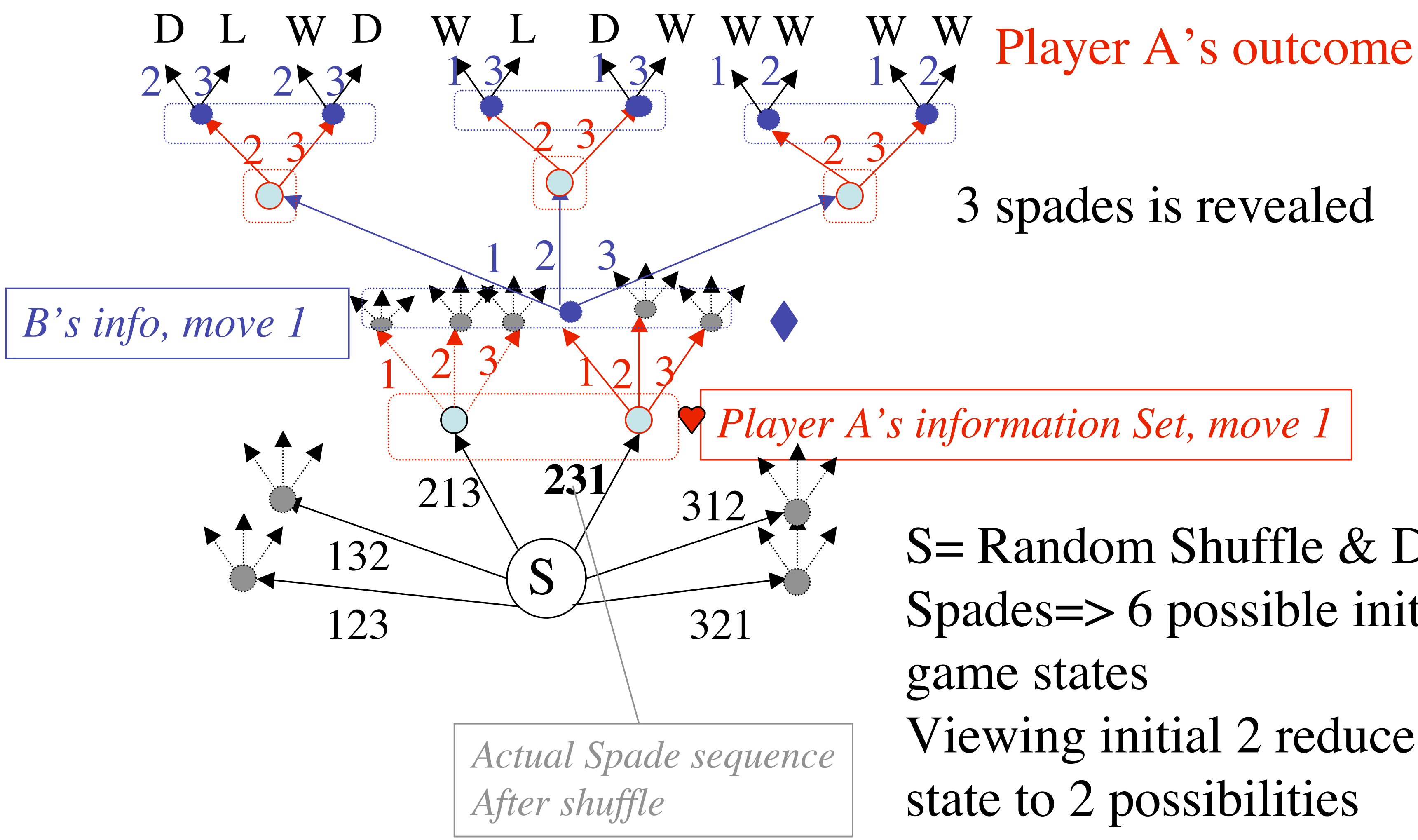
e.g. Actual play





What should B’s move Be?

Game Tree for 3-card Goofspiel, A's move hidden



S= Random Shuffle & Deal of Spades=> 6 possible initial game states

Viewing initial 2 reduces game state to 2 possibilities

Games in Normal Form

- Enumerate all possible strategies
  - Each strategy is a planned sequence moves, contingent on each information state.
  - Example:
    - A strategy: play Spade +1 (with 1 played for 3)
    - B strategy: match 1st spade, then play larger 2 remaining cards if A plays 3 first. Otherwise, play the smaller.

Deck	Player A	Player B	Cards Won by A	Outcome for A
123	231	123	1,2	Draw
132	213	123	1,draw on 2	Lose
213	321	231	2, draw on 3	Win
231	312	231	2,1	Draw
312	123	312	1,2	Draw
321	132	3122	2, draw on 1	Lose

Definition: normal-form or strategic-form representation

- The *normal-form* (*or strategic-form*) *representation* of a game *G* specifies:
  - ➢ A finite set of players $\{1, 2, ..., n\}$,
  - ➢ players' strategy spaces $S_1 \;\; S_2 \;\; ... \;\; S_n$ and
  - ➢ their payoff functions $u_1 \;\; u_2 \;\; ... \;\; u_n$
    where $u_i : S_1 \_ S_2 \_ ... \_ S_n \_ R$.

Games in Normal Form (2 player)

- Make a table with all pairs of event contingent strategies, and place in the cell the values of the outcomes for both players

A's \ B's	S_1	…..	S_N
S_1	$U_1(o_{11}),U_2(o_{11})$	…..	$U_1(o_{N1}),U_2(o_{N1})$
:	…..	…..	…..
S_M	$U_1(o_{1M}),U_2(o_{1M})$	…..	$U_1(o_{NM}),U_2(o_{NM})$

Normal-form representation: 2-player game

- Bi-matrix representation
  - 2 players: Player 1 and Player 2
  - Each player has a finite number of strategies
- Example:
  $S_1=\{s_{11}, \quad s_{12}, \quad s_{13}\}$ $\quad S_2=\{s_{21}, \quad s_{22}\}$
- (Outcomes of pairs of strategies assumed known)

Player 2

Player 1		s_{21}	s_{22}
	s_{11}	$u_1(s_{11}, s_{21})$, $u_2(s_{11}, s_{21})$	$u_1(s_{11}, s_{22})$, $u_2(s_{11}, s_{22})$
	s_{12}	$u_1(s_{12}, s_{21})$, $u_2(s_{12}, s_{21})$	$u_1(s_{12}, s_{22})$, $u_2(s_{12}, s_{22})$
	s_{13}	$u_1(s_{13}, s_{21})$, $u_2(s_{13}, s_{21})$	$u_1(s_{13}, s_{22})$, $u_2(s_{13}, s_{22})$

Classic Example: Prisoners' Dilemma

- Two suspects held in separate cells are charged with a major crime. However, there is not enough evidence.
- Both suspects are told the following policy:
  - ➢ If neither confesses then both will be convicted of a minor offense and sentenced to one month in jail.
  - ➢ If both confess then both will be sentenced to jail for six months.
  - ➢ If one confesses but the other does not, then the confessor will be released but the other will be sentenced to jail for nine months.

		Prisoner 2: Mum	Prisoner 2: Confess
Prisoner 1	Mum	-1 , -1	-9 , 0
	Confess	0 , -9	-6 , -6

Example: The battle of the sexes

- At the separate workplaces, Chris and Pat must choose to attend either an opera or a prize fight in the evening.
- Both Chris and Pat know the following:
  - Both would like to spend the evening together.
  - But Chris prefers the opera.
  - Pat prefers the prize fight.

		Pat: Opera	Pat: Prize Fight
Chris	Opera	2 , 1	0 , 0
	Prize Fight	0 , 0	1 , 2

Example: Matching pennies

- Each of the two players has a penny.
- Two players must simultaneously choose whether to show the Head or the Tail.
- Both players know the following rules:
  - If two pennies match (both heads or both tails) then player 2 wins player 1's penny.
  - Otherwise, player 1 wins player 2's penny.

		Player 2 Head	Player 2 Tail
Player 1	Head	-1 , 1	1 , -1
	Tail	1 , -1	-1 , 1

Static (or simultaneous-move) games of complete information

A static (or simultaneous-move) game consists of:

- A set of players (at least two players)
- For each player, a set of strategies/actions
- Payoffs received by each player for the combinations of the strategies, or for each player, preferences over the combinations of the strategies

- {Player 1, Player 2, ... Player n}
- $\boldsymbol{S_1 \ \ S_2 \ \ldots \ S_n}$
- $\boldsymbol{u_i(s_1, s_2, \ldots s_n)}$, for all $\boldsymbol{s_1 \in S_1, s_2 \in S_2, \ldots s_n \in S_n}$.

Static (or simultaneous-move) games of complete information

- Simultaneous-move
  - ➢ Each player chooses his/her strategy without knowledge of others' choices.
- Complete information
  - ➢ Each player's strategies and payoff function are common knowledge among all the players.
- Assumptions on the players
  - ➢ Rationality
    - Players aim to maximize their payoffs
    - Players are perfect calculators
  - ➢ Each player knows that other players are rational

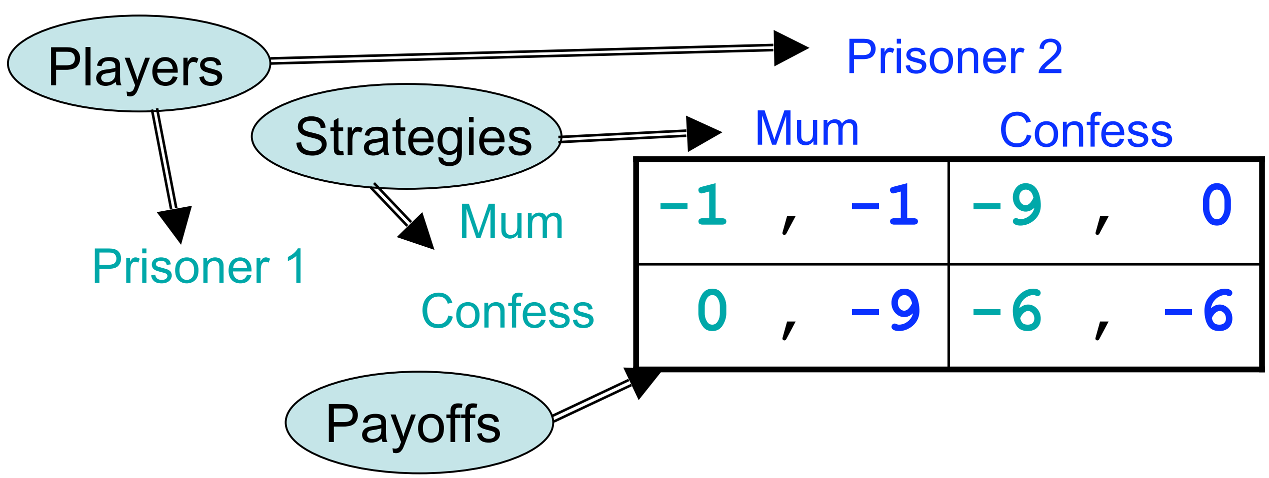
Static (or simultaneous-move) games of complete information

- The players cooperate?
  - No. Only noncooperative games
- The timing
  - Each player i chooses his/her strategy s_i without knowledge of others' choices.
  - Then each player i receives his/her payoff $u_i(s_1, s_2, ..., s_n)$.
  - The game ends.

Classic example: Prisoners' Dilemma: normal-form representation

- Set of players: {Prisoner 1, Prisoner 2}
- Sets of strategies: $S_1 = S_2 =$ {**<u>M</u>um, <u>C</u>onfess**}
- Payoff functions:

u_1(M, M)=-1, u_1(M, C)=-9, u_1(C, M)=0, u_1(C, C)=-6;
u_2(M, M)=-1, u_2(M, C)=0, u_2(C, M)=-9, u_2(C, C)=-6



Prisoner 1 \ Prisoner 2	Mum	Confess
Mum	-1 , -1	-9 , 0
Confess	0 , -9	-6 , -6

Example: The battle of the sexes

		Pat: Opera	Pat: Prize Fight
Chris	Opera	2 , 1	0 , 0
	Prize Fight	0 , 0	1 , 2

- Normal (or strategic) form representation:
  - Set of players: { Chris, Pat } (={Player 1, Player 2})
  - Sets of strategies: $S_1 = S_2$ = { Opera, Prize Fight}
  - Payoff functions:

    u_1(O, O)=2, u_1(O, F)=0, u_1(F, O)=0, u_1(F, O)=1;
    u_2(O, O)=1, u_2(O, F)=0, u_2(F, O)=0, u_2(F, F)=2

Example: Matching pennies

		Player 2 Head	Player 2 Tail
Player 1	Head	-1 , 1	1 , -1
	Tail	1 , -1	-1 , 1

- Normal (or strategic) form representation:
  - Set of players: {Player 1, Player 2}
  - Sets of strategies: $\boldsymbol{S}_1 = \boldsymbol{S}_2 = \{ \underline{\text{H}}\text{ead}, \underline{\text{T}}\text{ail} \}$
  - Payoff functions:

u_1(H, H)=-1, u_1(H, T)=1, u_1(T, H)=1, u_1(H, T)=-1;
u_2(H, H)=1, u_2(H, T)=-1, u_2(T, H)=-1, u_2(T, T)=1

Games for eliciting social preferences

Table 1: Seven experimental games useful for measuring social preferences

Game	Definition of the Game	Real life Example	Predictions with rational and selfish players	Experimental regularities, References	Interpretation
Prisoners' dilemma Game	Two players, each of whom can either cooperate or defect. Payoffs are as follows: Cooperate / Defect Cooperate: H,H / S,T Defect: T,S / L,L H>L, T>H, L>S	Production of negative externalities (pollution, loud noise), exchange without binding contracts, status competition.	Defect	50% choose Cooperate. Communication increases frequency of cooperation Dawes (1980)**	Reciprocate expected cooperation
Public Goods Game	n players simultaneously decide about their contribution g_i. ($0 \le g_i \le y$) where y is players' endowment; each player *i* earns $\pi_i = y - g_i + mG$ where G is the sum of all contributions and $m<1<mn$.	Team compensation, cooperative production in simple societies, overuse of common resources (e.g., water, fishing grounds)	Each player contributes nothing, i.e. $g_i = 0$.	Players contribute 50% of y in the one-shot game. Contributions unravel over time. Majority chooses $g_i=0$ in final period. Communication strongly increases cooperation. Individual punishment opportunities greatly increase contributions. Ledyard (1995)**.	Reciprocate expected cooperation
Ultimatum Game	Division of a fixed sum of money S between a Proposer and a Responder. Proposer offers x. If Responder rejects x both earn zero, if x is accepted the Proposer earns S – x and the Responder earns x.	Monopoly pricing of a perishable good; "11th-hour" settlement offers before a time deadline	Offer $x=\varepsilon$; where ε is the smallest money unit. Any $x>0$ is accepted.	Most offers are between .3 and .5S. x <.2S rejected half the time. Competition among Proposers has a strong x-increasing effect; competition among Responders strongly decreases x. Güth et al (1982)*, Camerer (2003)**	Responders punish unfair offers; negative reciprocity
Dictator Game	Like the ultimatum game but the Responder cannot reject, i.e., the "Proposer" dictates (S-x, x).	Charitable sharing of a windfall gain (lottery winners giving anony-mously to strangers)	No sharing, i.e., x = 0	On average "Proposers" allocate x=.2S. Strong variations across experiments and across individuals Kahneman et al (1986)*, Camerer (2003)**	Pure altruism

More Games

Trust Game	Investor has endowment S and makes a transfer y between 0 and S to the Trustee. Trustee receives 3y and can send back any x between 0 and 3y. Investor earns S – y + x. Trustee earns 3y – x.	Sequential exchange without binding contracts (buying from sellers on Ebay)	Trustee repays nothing: x = 0. Investor invests nothing: y = 0.	On average y = .5S and trustees repay slightly less than .5S. x is increasing in y. Berg et al (1995)*, Camerer (2003)**	Trustees show positive reciprocity.
Gift Exchange Game	"Employer" offers a wage *w* to the "worker" and announces a desired effort level $\hat{e}$. If worker rejects (w, $\hat{e}$) both earn nothing. If worker accepts, he can choose *any e* between 1 and 10. Then employer earns 10e – w and worker earn $w - c(e)$. $c(e)$ is the effort cost which is strictly increasing in *e*.	Noncontractibility or nonenforceability of the performance (effort, quality of goods) of workers or sellers.	Worker chooses e = 1. Employer pays the minimum wage.	Effort increases with the wage *w*. Employers pay wages that are far above the minimum. Workers accept offers with low wages but respond with e = 1. In contrast to the ultimatum game competition among workers (i.e., Responders) has no impact on wage offers. Fehr et al (1993)*	Workers reciprocate generous wage offers. Employers appeal to workers' reciprocity by offering generous wages.
Third Party Punishment Game	A and B play a dictator game. C observes how much of amount S is allocated to B. C can punish A but the punishment is also costly for C.	Social disapproval of unacceptable treatment of others (scolding neighbors).	A allocates nothing to B. C never punishes A.	Punishment of A is the higher the less A allocates to B. Fehr and Fischbacher (2001a)*	C sanctions violation of a sharing norm.

Note: ** denotes survey papers, * denotes papers that introduced the respective games.

Core Concepts we Need from Game Theory

- *Strategy*
- *Mixed strategy*
- *Information set*
- Dominance
- Nash Equilibrium
- Subgame Perfection
- Types of Players (Bayesian games)

"Understanding" a Game

Fundamental assumption of game theory:

Get Rid of the Strictly Dominated strategies. They Won't Happen.

	C	D
C	-1 , -1	-9 , 0
D	0 , -9	-6 , -6

In some cases (e.g. prisoner's dilemma) this means, if players are "rational" we can predict the outcome of the game.

Definition: strictly dominated strategy

In the normal-form game $\{S_1, S_2, \ldots, S_n, u_1, u_2, \ldots, u_n\}$, let $s_i', s_i'' \in S_i$ be feasible strategies for player i. Strategy s_i' is ***strictly dominated*** by strategy s_i'' if

$$\boxed{\begin{array}{c} u_i(s_1, s_2, \ldots s_{i-1}, s_i', s_{i+1}, \ldots, s_n) \\ < u_i(s_1, s_2, \ldots s_{i-1}, s_i'', s_{i+1}, \ldots, s_n) \end{array}}$$

s_i'' is strictly better than s_i'

$$\boxed{\text{for all } s_1 \in S_1, s_2 \in S_2, \ldots, s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, \ldots, s_n \in S_n.}$$

regardless of other players' choices

		Prisoner 2: Mum	Prisoner 2: Confess
Prisoner 1	Mum	-1 , -1	-9 , 0
	Confess	0 , -9	-6 , -6

Definition: weakly dominated strategy

In the normal-form game $\{S_1, S_2, \ldots, S_n, u_1, u_2, \ldots, u_n\}$, let $s_i', s_i'' \in S_i$ be feasible strategies for player i. Strategy s_i' is ***weakly dominated*** by strategy s_i'' if

$$\boxed{\begin{array}{c} u_i(s_1, s_2, \ldots s_{i-1}, s_i', s_{i+1}, \ldots, s_n) \\ \leq \text{(but not always =) } u_i(s_1, s_2, \ldots s_{i-1}, s_i'', s_{i+1}, \ldots, s_n) \end{array}}$$

s_i'' is at least as good as s_i'

$$\boxed{\text{for all } s_1 \in S_1, s_2 \in S_2, \ldots, s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, \ldots, s_n \in S_n.}$$

regardless of other players' choices

		Player 2: L	Player 2: R
Player 1	U	1 , 1	2 , 0
	B	0 , 2	2 , 2

Strictly and weakly dominated strategy

- A rational player never chooses a strictly dominated strategy (that it perceives). Hence, any strictly dominated strategy can be eliminated.
- A rational player may choose a weakly dominated strategy.

"Understanding" a Game

Fundamental assumption of game theory:

Get Rid of the Strictly Dominated strategies. They Won't Happen.

	C	D
C	~~-1 , -1~~	~~-9 , 0~~
D	0 , -9	-6 , -6

In some cases (e.g. prisoner's dilemma) this means, if players are "rational" we can predict the outcome of the game.

"Understanding" a Game

Fundamental assumption of game theory:

Get Rid of the Strictly Dominated strategies. They Won't Happen.

	C	D
C	-1 , -1	-9 , 0
D	0 , -9	-6 , -6

→

	C	D
D	0 , -9	-6 , -6

In some cases (e.g. prisoner's dilemma) this means, if players are "rational" we can predict the outcome of the game.

"Understanding" a Game

Fundamental assumption of game theory:

Get Rid of the Strictly Dominated strategies. They Won't Happen.

	C	D
C	-1 , -1	-9 , 0
D	0 , -9	-6 , -6

→

	C	D
D	~~0 , -9~~	-6 , -6

In some cases (e.g. prisoner's dilemma) this means, if players are "rational" we can predict the outcome of the game.

Strict Domination Removal Example

Player A \ Player B	I	II	III	IV
I	3 , 1	4 , 1	5 , 9	2 , 6
II	5 , 3	5 , 8	9 , 7	9 , 3
III	2 , 3	8 , 4	6 , 2	6 , 3
IV	3 , 8	3 , 1	2 , 3	4 , 5

So is strict domination the best tool for predicting what will transpire in a game ?

Strict Domination doesn't capture the whole picture

	I	II	III
I	0 , 4	4 , 0	5 , 3
II	4 , 0	0 , 4	5 , 3
III	3 , 5	3 , 5	6 , 6

What strict domination eliminations can we do?

What would <u>you</u> predict the players of this game would do?

Nash Equilibria

$$S_1^* \in S_1, S_2^* \in S_2, \cdots S_n^* \in S_n$$

are a NASH EQUILIBRIUM iff

$$\forall i \quad S_i^* = \arg\max_{S_i} u_i\left(S_1^*, S_2^*, \cdots S_{i-1}^*, S_i, S_{i+1}^* \cdots S_n^*\right)$$

	I_b	II_b	III_b
I_a	0 4	4 0	5 3
II_a	4 0	0 4	5 3
III_a	3 5	3 5	6 6

(III_a, III_b) is a N.E. because

$$u_1(\mathrm{III_a}, \mathrm{III_b}) = \max \begin{bmatrix} u_1(\mathrm{I_a}, \mathrm{III_b}) \\ u_1(\mathrm{II_a}, \mathrm{III_b}) \\ u_1(\mathrm{III_a}, \mathrm{III_b}) \end{bmatrix}$$

$$\text{AND } u_2(\mathrm{III_a}, \mathrm{III_b}) = \max \begin{bmatrix} u_2(\mathrm{III_a}, \mathrm{I_b}) \\ u_2(\mathrm{III_a}, \mathrm{II_b}) \\ u_2(\mathrm{III_a}, \mathrm{III_b}) \end{bmatrix}$$

- If (S_1^*, S_2^*) is an N.E. then player 1 won't want to change their play given player 2 is doing S_2^*
- If (S_1^*, S_2^*) is an N.E. then player 2 won't want to change their play given player 1 is doing S_1^*

Find the NEs:

-1 -1	-9 0
0 -9	-6 -6

0 4	4 0	5 3
4 0	0 4	5 3
3 5	3 5	6 6

- Is there always at least one NE ?
- Can there be more than one NE ?

Example with no NEs among the pure strategies:

	S_1	S_2
S_1	0 1	1 0
S_2	1 0	0 1

2-player mixed strategy Nash Equilibrium

The pair of mixed strategies (M_A , M_B) are a Nash Equilibrium iff

- M_A is player A's best mixed strategy response to M_B

AND

- M_B is player B's best mixed strategy response to M_A

Fundamental Theorems

- In the n-player pure strategy game $G=\{S_1\ S_2 \cdots S_n;\ u_1\ u_2 \cdots u_n\}$, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(S_1^*, S_2^* \cdots S_n^*)$ then these strategies are the unique NE of the game
- Any NE will survive iterated elimination of strictly dominated strategies
- [Nash, 1950] If *n* is finite and S_i is finite $\forall i$, then there exists at least one NE (possibly involving mixed strategies)

Back to the Battle

		Patricia: Opera	Patricia: Prize Fight
Chris	Opera	2 , 1	-1 , -1
	Prize Fight	-1 , -1	1 , 2

Two Nash Equilibria

Payoffs

$$M_1 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

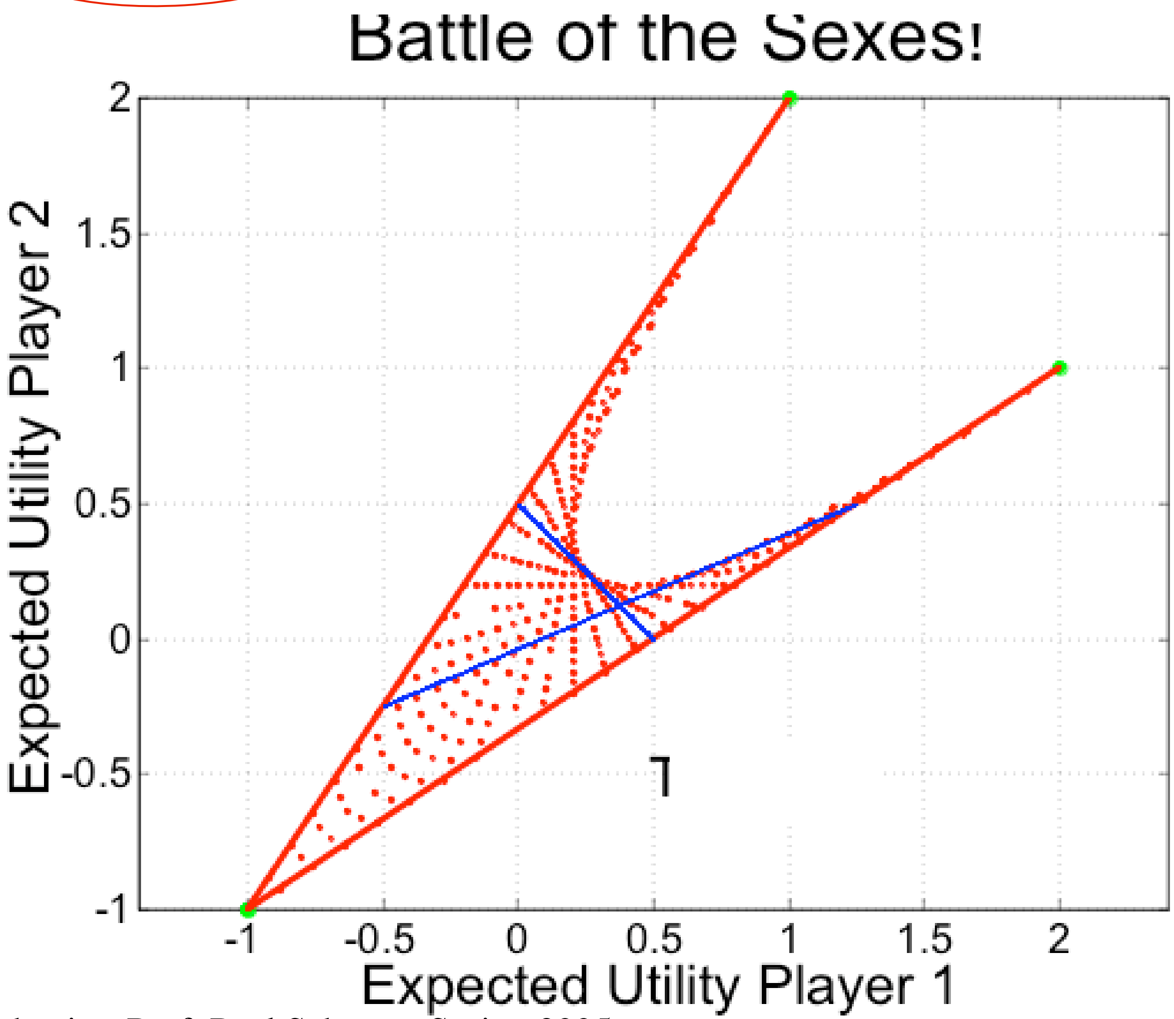
Mixed strategies

$$p_1 = \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}, p_2 = \begin{bmatrix} \beta \\ 1-\beta \end{bmatrix}$$

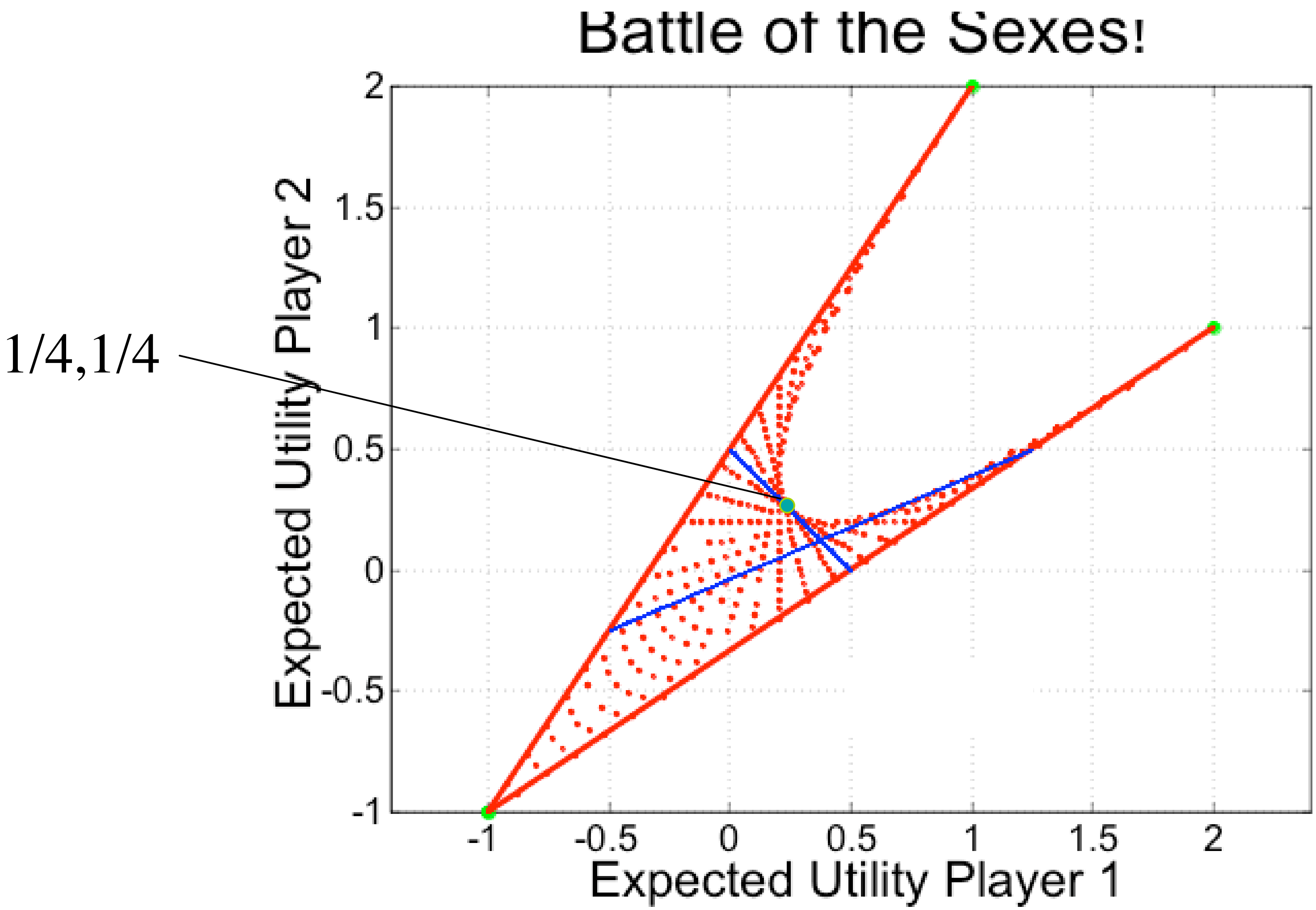
Values

$$u_1 = p_1^T M_1 p_2$$

$$u_2 = p_1^T M_2 p_2$$

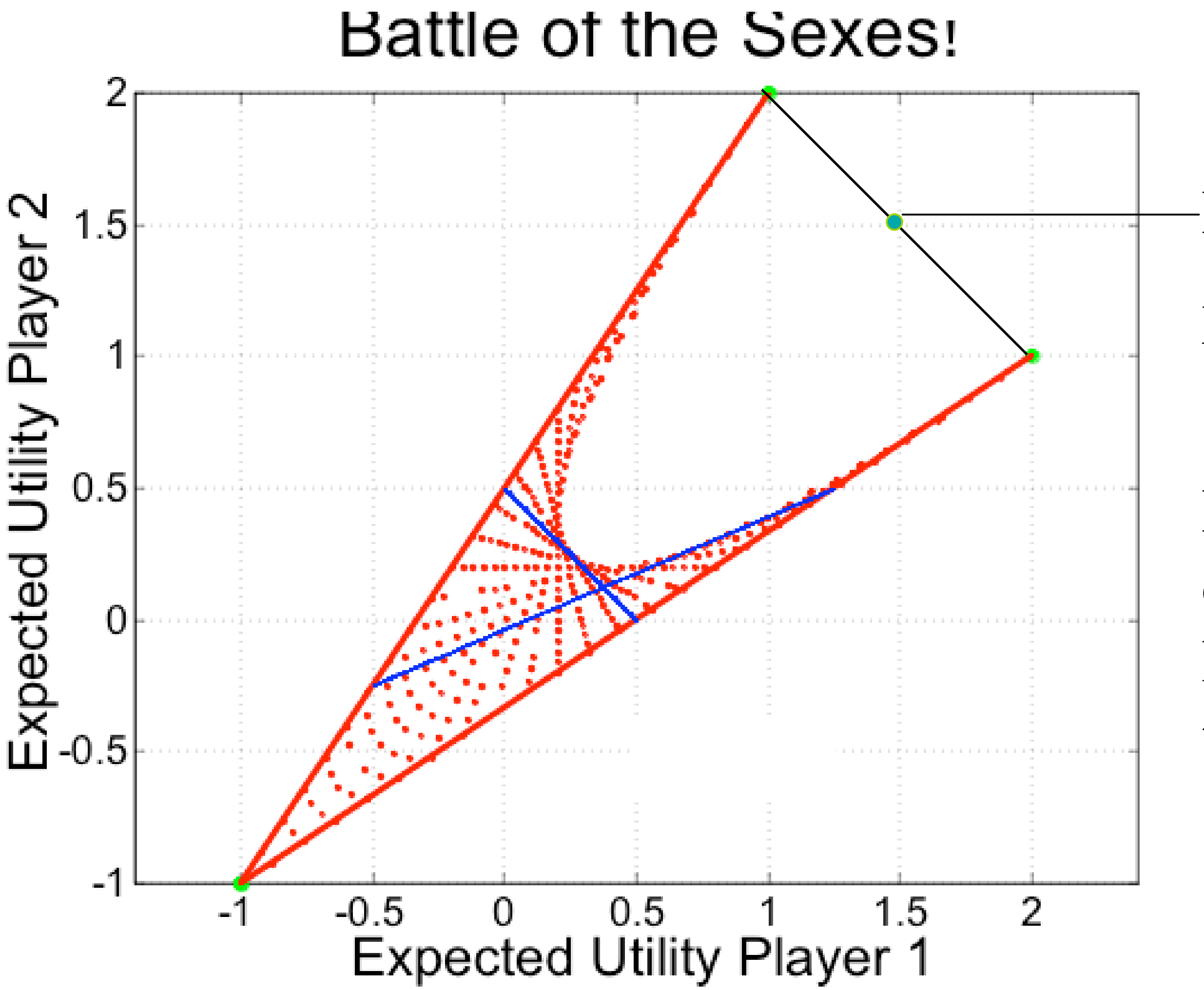


What is Fair?



Battle of the Sexes!
1/4,1/4
Expected Utility Player 2
Expected Utility Player 1
2
1.5
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.5
2

Bargaining- Agreeing to Eliminate strategy pairs



Fair-Flip a coin and Agree to let coin-flip be binding.

Requires a coordinated decision- Chris and Pat have to talk to achieve this.

Nash Equilibria Being Useful

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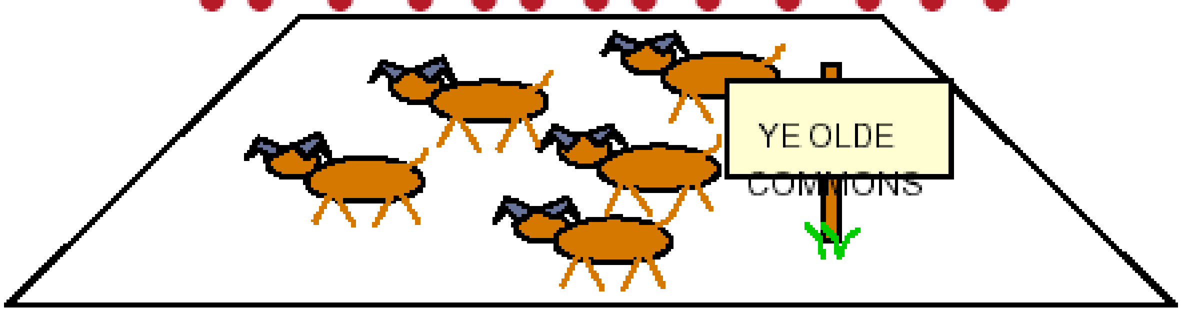
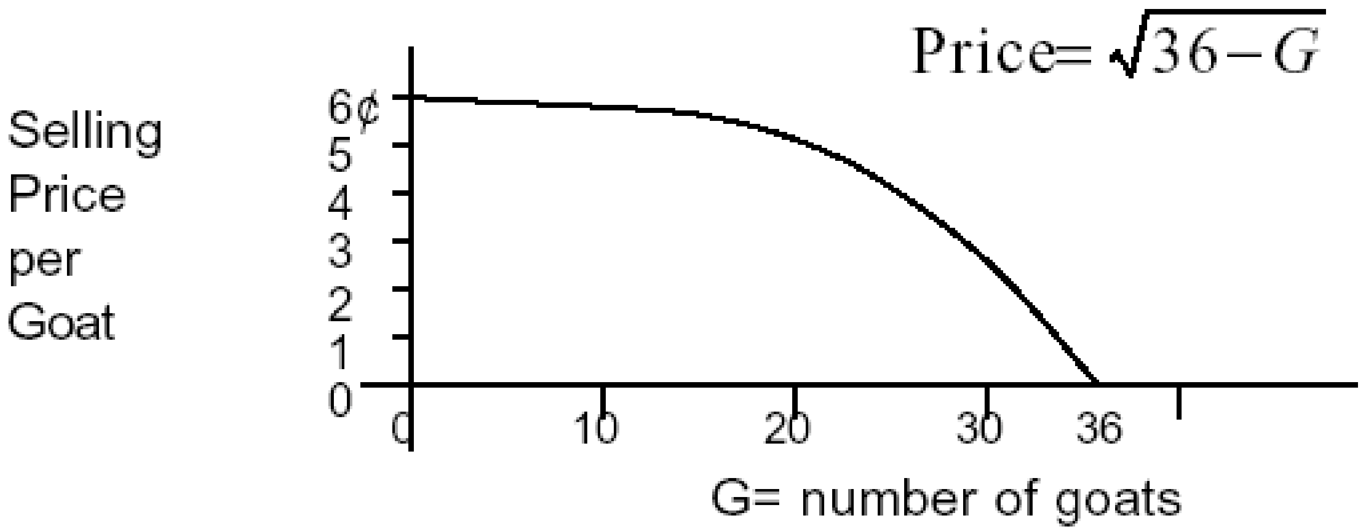
Commons…



- You graze goats on the commons to eventually fatten up and sell
- The more goats you graze the less well fed they are
- And so the less money you get when you sell them

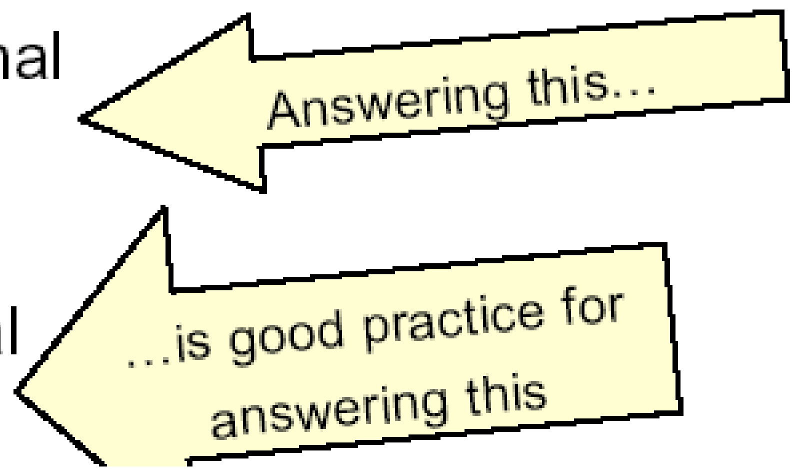
Commons Facts



How many goats would one rational farmer choose to graze?

What would the farmer earn?

What about a group of *n* individual farmers?



n farmers

i'th farmer has an infinite space of strategies

$$g_i \in [\,0\,,\,36\,]$$

An outcome of

$$(\,g_1\,,\,g_2\,,\,g_3 \cdots,\,g_n\,)$$

will pay how much to the i'th farmer?

$$g_i \times \sqrt{36 - \sum_{j=1}^{n} g_j}$$

Let's Assume a pure Nash Equilibrium exists.

Call it

$$\left(g_1^*, g_2^*, \cdots g_n^*\right)$$

What can we say about g_1^* ?

$$g_i^* = \arg\max_{g_i} \begin{bmatrix} \text{Payoff to farmer i, assuming} \\ \text{the other players play} \\ \left(g_1^*, g_2^*, \cdots g_{i-1}^*, g_{i+1}^*, \cdots g_n^*\right) \end{bmatrix}$$

For Notational Convenienc e,

write $G_{-i}^* = \sum_{j \neq i} g_j^*$

THEN

$$g_i^* = \arg\max_{g_i} \left[g_i \sqrt{36 - g_i - G_{-i}^*} \right]$$

Let's Assume a pure Nash Equilibrium exists.

Call it

$$\left(g_1^*, g_2^*, \cdots g_n^*\right)$$

What can we say about

$$g_i^* = \arg\max_{g_i} \left[\begin{array}{l} \text{Payo} \\ \text{the c} \\ \left(g_1^*, \right. \end{array} \right.$$

For Notational Conven

write $\quad G_{-i}^* = \sum_{j \neq i} g_j^*$

THEN

$$g_i^* = \arg\max_{g_i} \left[g_i \sqrt{36 - g_i - G_{-i}^*} \right]$$

g^*_i must satisfy

$$\frac{\partial}{\partial g_i^*} g_i^* \sqrt{36 - g_i^* - G_{-i}^*} = 0$$

therefore

$$\frac{36 - G_{-i}^* - \frac{3}{2} g_i^*}{\sqrt{36 - g_i^* - G_{-i}^*}} = 0$$

We have *n* linear equations in *n* unknowns

$$g_1^* = 24 - 2/3(\quad g_2^* + g_3^* + \quad \cdots g_n^*)$$
$$g_2^* = 24 - 2/3(g_1^* + \quad g_3^* + \quad \cdots g_n^*)$$
$$g_3^* = 24 - 2/3(g_1^* + g_2^* + \quad g_4^* \cdots g_n^*)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$g_n^* = 24 - 2/3(g_1^* + \quad \cdots g_{n-1}^*)$$

Clearly all the g_i^*'s are the same (Proof by "it's bloody obvious")

Write $g^* = g_1^* = \cdots g_n^*$

Solution to $g^* = 24 - 2/3(n-1)g^*$ is: $g^* = \dfrac{72}{2n+1}$

Consequences

At the Nash Equilibrium a rational farmer grazes

$\frac{72}{2n+1}$ goats.

How many goats in general will be grazed? Trivial algebra gives: $36 - \frac{36}{2n+1}$ goats total being grazed

[as n --> *infinity* , #goats --> 36]

How much profit per farmer? $\frac{432}{(2n+1)^{3/2}}$ ← 1.26¢ if 24 farmers

How much if the farmers could all cooperate?

$$\frac{24 \cdot \mathrm{sqrt}(12)}{n} = \frac{83.1}{n}$$ ← 3.46¢ if 24 farmers

The Tragedy

The farmers act “rationally” and earn 1.26 cents each.

But if they’d all just got together and decided “one goat each” they’d have got 3.46 cents each.

Is there a bug in Game Theory?
in the Farmers?
in Nash?

Would you recommend the farmers hire a police force?

Less Tragic with Repeated Plays?

- Does the Tragedy of the Commons matter to us when we're analyzing human behavior?
- Maybe repeated play means we can learn to cooperate??

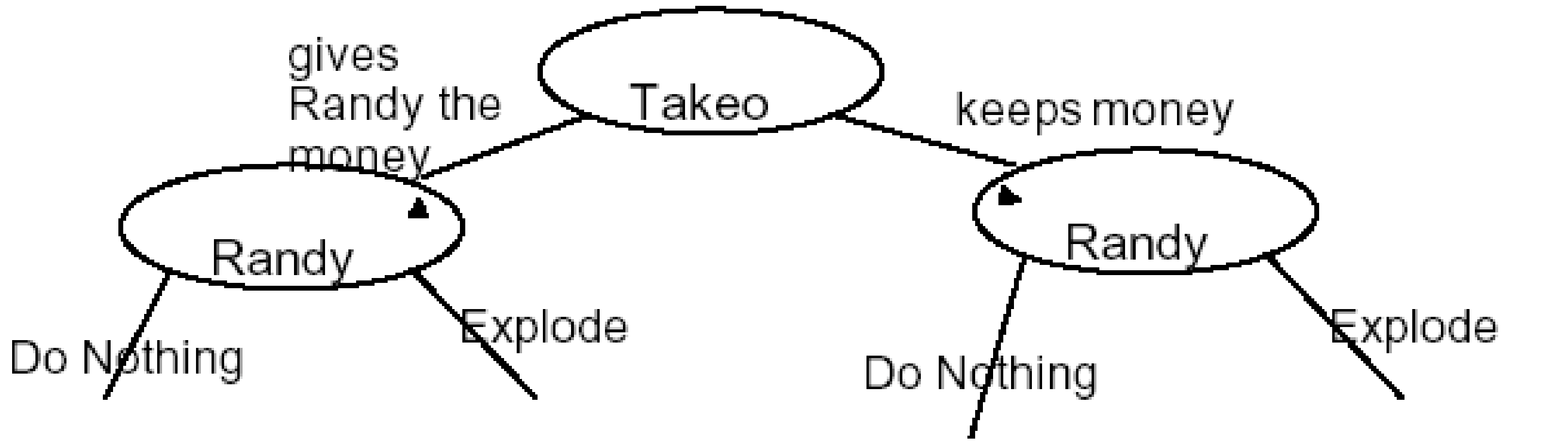
Repeated Games with Implausible Threats

Takeo and Randy are stuck in an elevator

Takeo has a $1000 bill

Randy has a stick of dynamite

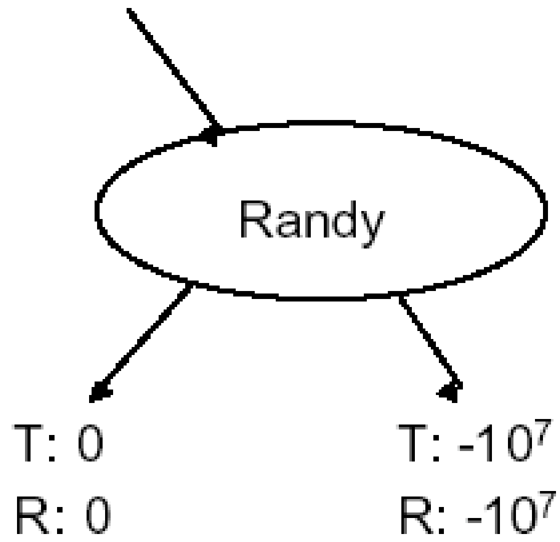
Randy says “Give me $1000 or I’ll blow us both up.”



Takeo:	-1000	Takeo:	-10^7	Takeo:	0	Takeo:	-10^7
Randy:	1000	Randy:	-10^7	Randy:	0	Randy:	-10^7

What should Takeo do?????

Using the formalism of Repeated Games With Implausible Threats, Takeo should Not give the money to Randy



Takeo Assumes Randy is Rational

At this node, Randy will choose the left branch

Repeated Games

Suppose you have a game which you are going to play a finite number of times.

What should you do?

2-Step Prisoner's Dilemma

GAME 1

Player A \ Player B	C	D
C	-1 , -1	-9 , 0
D	0 , -9	-6 , -6

GAME 2
(Played with knowledge of outcome of GAME 1)

Player A \ Player B	C	D
C	-1 , -1	-9 , 0
D	0 , -9	-6 , -6

Idea 1

Player A has four pure strategies

C then C
C then D
D then C
D then D

Ditto for B

Is Idea 1 correct?

Important Theoretical Result:

Assuming Implausible Threats, if the game G has a unique N.E. $(s_1^*, \cdots s_n^*)$ then the new game of repeating G T times, and adding payouts, has a unique N.E. of repeatedly choosing the original N.E. $(s_1^*, \cdots s_n^*)$ in every game.

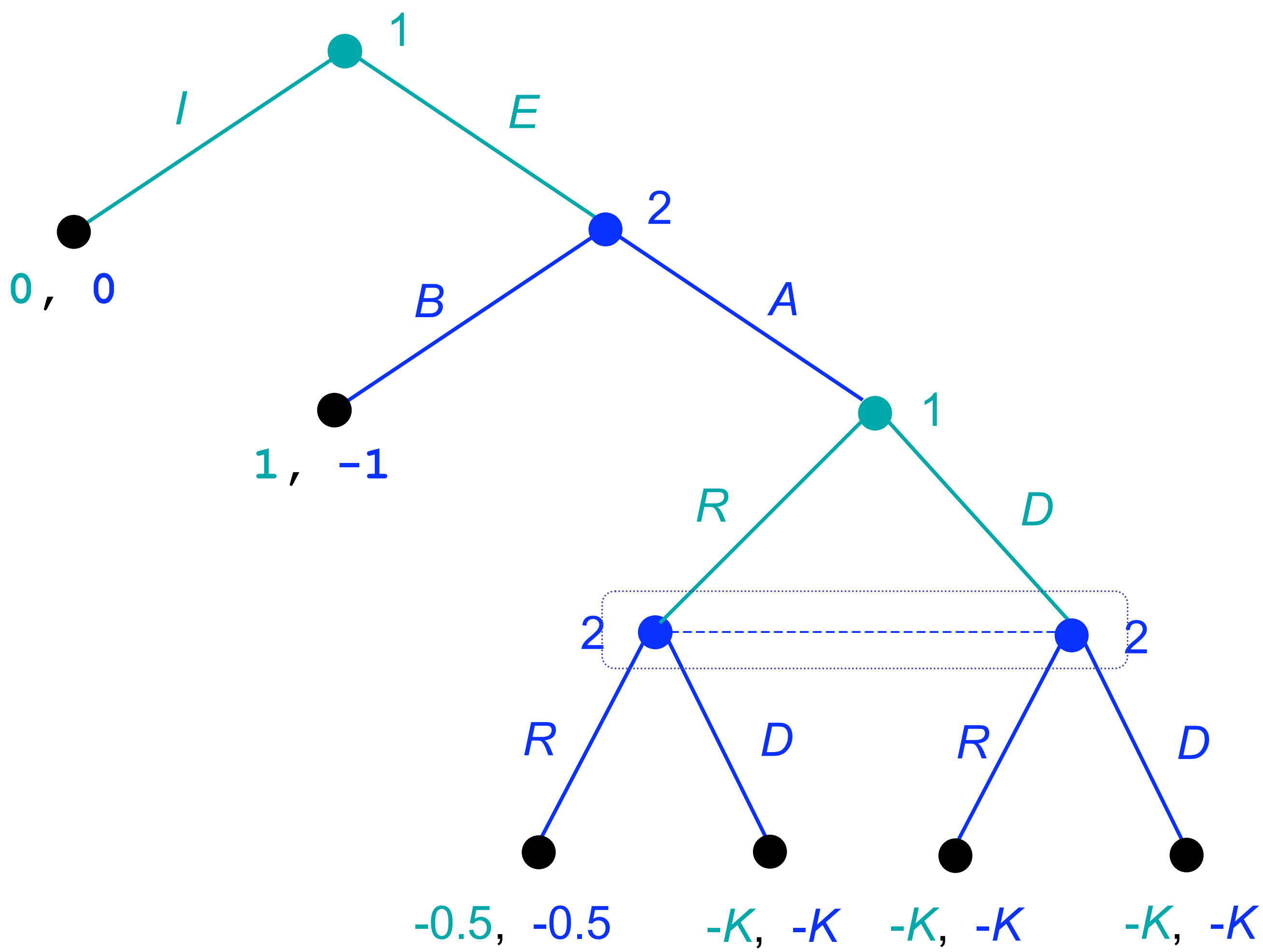
If you're about to play prisoner's dilemma 20 times, you should defect 20 times.

DRAT ☹

Example: mutually assured destruction

- Two superpowers, 1 and 2, have engaged in a provocative incident. The timing is as follows.
- The game starts with superpower 1's choice either ignore the incident (*I*), resulting in the payoffs (0, 0), or to escalate the situation (*E*).
- Following escalation by superpower 1, superpower 2 can back down (*B*), causing it to lose face and result in the payoffs (1, -1), or it can choose to proceed to an atomic confrontation situation (*A*). Upon this choice, the two superpowers play the following simultaneous move game.
- They can either retreat (*R*) or choose to doomsday (*D*) in which the world is destroyed. If both choose to retreat then they suffer a small loss and payoffs are (-0.5, -0.5). If either chooses doomsday then the world is destroyed and payoffs are ($-K$, $-K$), where K is very large number.

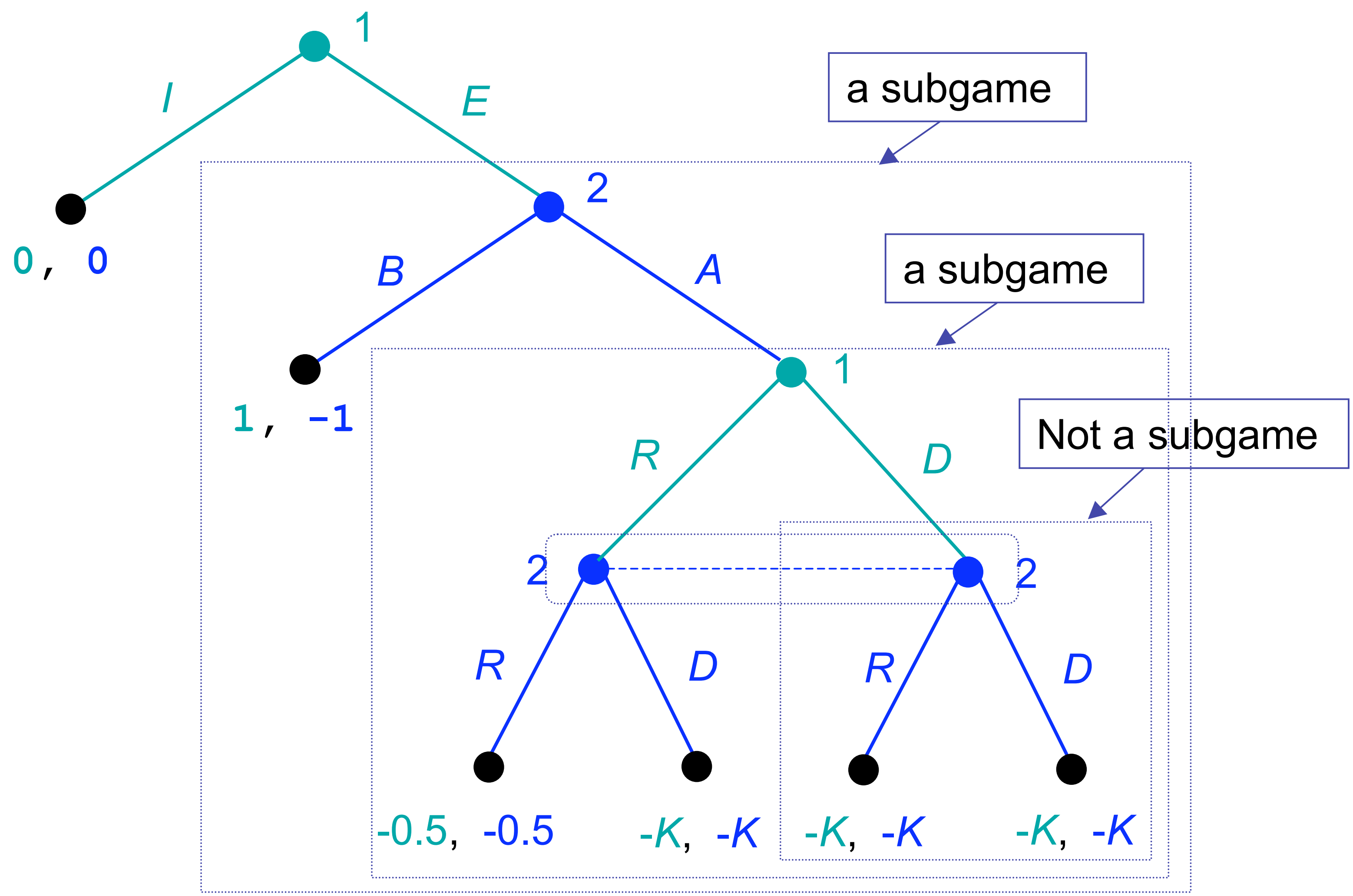
Example: mutually assured destruction



Subgame

- A subgame of a dynamic game tree
  - begins at a singleton information set (an information set contains a single node), and
  - includes all the nodes and edges following the singleton information set, and
  - does not cut any information set; that is, if a node of an information set belongs to this subgame then all the nodes of the information set also belong to the subgame.

Subgame: illustration

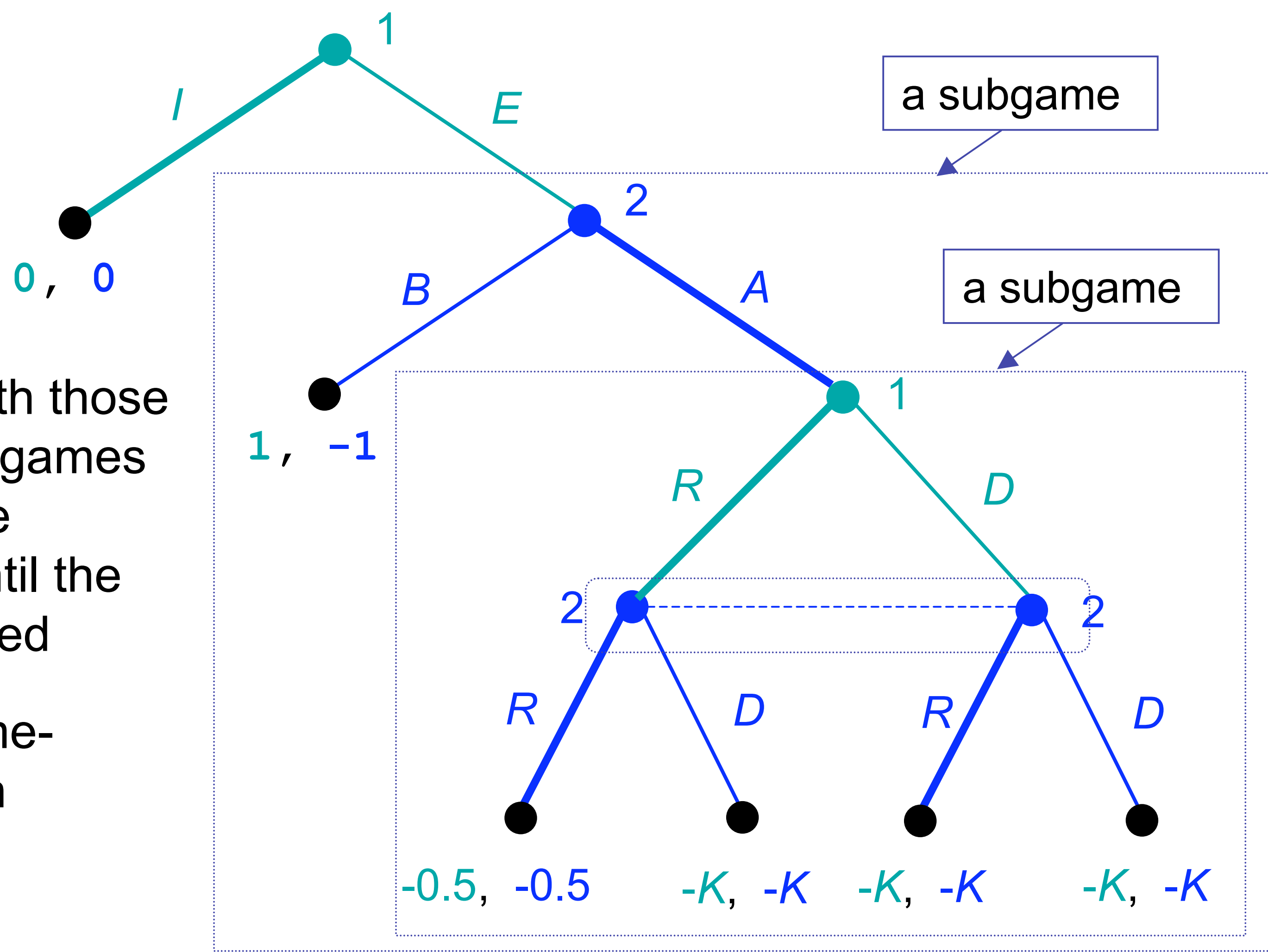


Subgame-perfect Nash equilibrium

- A Nash equilibrium of a dynamic game is subgame-perfect if the strategies of the Nash equilibrium constitute or induce a Nash equilibrium in every subgame of the game.

- Subgame-perfect Nash equilibrium is a Nash equilibrium.

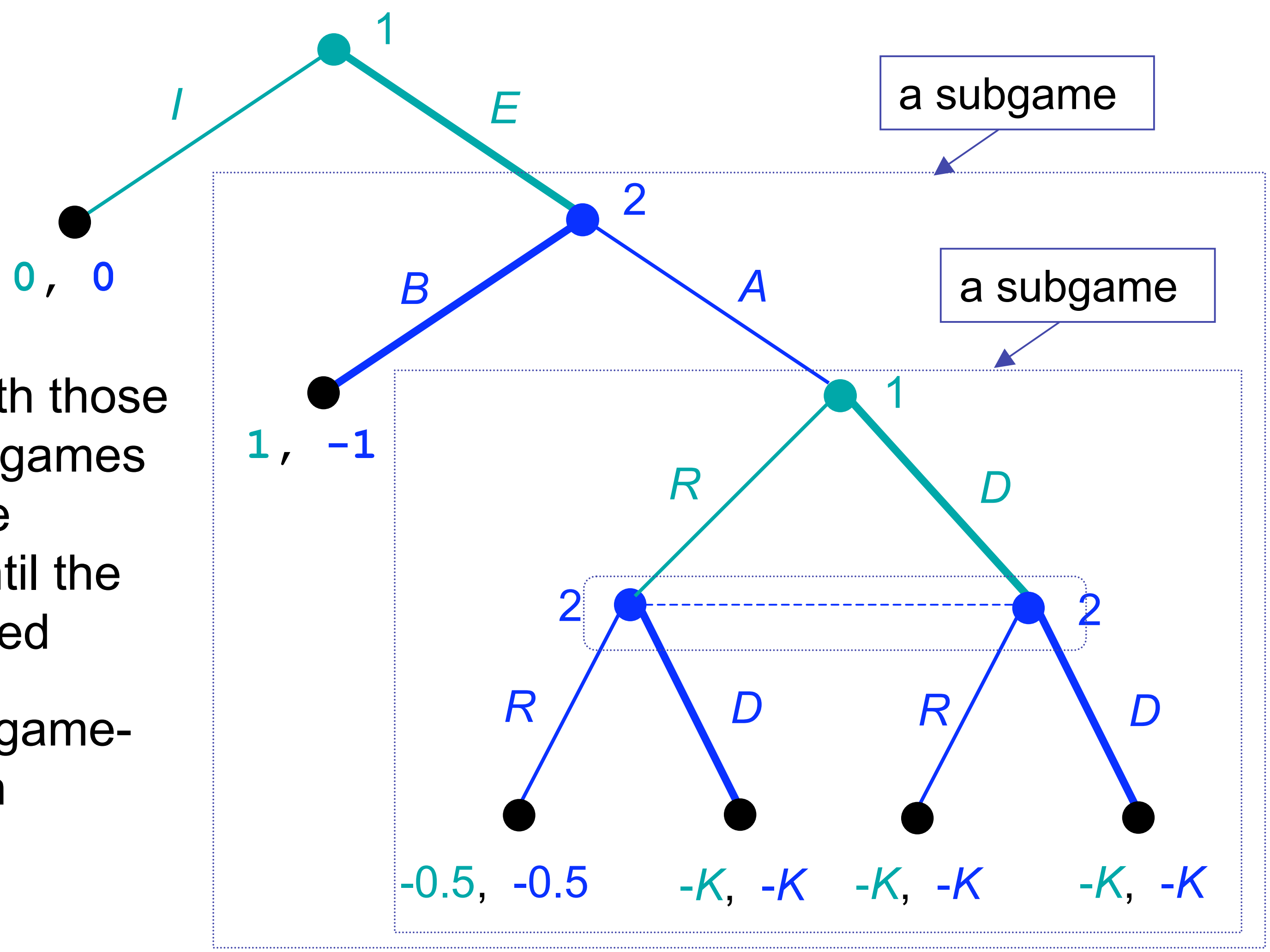
Find subgame perfect Nash equilibria: backward induction



- Starting with those smallest subgames
- Then move backward until the root is reached

One subgame-perfect Nash equilibrium (*IR*, *AR*)

Find subgame perfect Nash equilibria: backward induction



- Starting with those smallest subgames
- Then move backward until the root is reached

Another subgame-perfect Nash equilibrium
(*ED*, *BD*)

Bayesian Games

You are Player A in the following game. What should you do?

		Player B	
		S_1	S_2
Player A	S_1	3 ?	-2 ?
	S_2	0 ?	6 ?

Question: When does this situation arise?

Recipe for Nash-Equilibrium-Based Analysis of Such Games

- Assume you've been given a problem where the i'th player chooses a real number x_i
- Guess the existence of a Nash equilibrium

$$(x_1^* , x_2^* \cdots x_n^*)$$

- Note that , $\forall$i,

$$x_i^* = \arg\max_{x_i} \left[\begin{array}{l} \text{Payoff to player } i \text{ if player } i \\ \text{plays "} x_i \text{" and the } j\text{'th player} \\ \text{plays } x_j^* \text{ for } j \neq i \end{array} \right]$$

- Hack the algebra, often using "at x_i* we have

Hockey lovers get 2 units for watching hockey, and 1 unit for watching football.

Football lovers get 2 units for watching football, and 1 unit for watching hockey.

Pat's a hockey lover.

Pat thinks Chris is probably a hockey lover also, but Pat is not sure.

Pat \ Chris	H	F
H	2 2	0 0
F	0 0	1 1

With 2/3 chance

Pat \ Chris	H	F
H	2 1	0 0
F	0 0	1 2

1/3 chance

In a Bayesian Game each player is given a type. All players know their own types but only a prob. dist. for their opponent's types

An n-player Bayesian Game has

a set of action spaces	$A_1 \cdot\cdot\ A_n$
a set of type spaces	$T_1 \cdot\cdot\ T_n$
a set of beliefs	$P_1 \cdot\cdot\ P_n$
a set of payoff functions	$u_1 \cdot\cdot\ u_n$

$P_{-i}(t_{-i}|t_i)$ is the prob dist of the types for the other players, given player i has type i .

$u_i(a_1 , a_2 \cdots a_n , t_i)$ is the payout to player i if player j chooses action a_j (with $a_j \in A_j$) (forall j=1,2,$\cdots n$) and if player i has type $t_i \in T_i$

Bayesian Games: Who Knows What?

We assume that all players enter knowing the full information about the A_i's , T_i's, P_i's and u_i's

The i'th player knows t_i, but not $t_1\ t_2\ t_3 \cdots t_{i-1}\ t_{i+1} \cdots t_n$

All players know that all other players know the above

And they know that they know that they know, *ad infinitum*

Definition: A strategy $S_i(t_i)$ in a Bayesian Game is a mapping from T_i? A_i : a specification of what action would be taken for each type

Example

$A_1 = \{H,F\}$ $A_2 = \{H,F\}$

$T_1 = \{\text{H-love},\text{Flove}\}$ $T_2 = \{\text{Hlove}, \text{Flove}\}$

$P_1\ (t_2 = \text{Hlove} \mid t_1 = \text{Hlove}) = 2/3$
$P_1\ (t_2 = \text{Flove} \mid t_1 = \text{Hlove}) = 1/3$
$P_1\ (t_2 = \text{Hlove} \mid t_1 = \text{Flove}) = 2/3$
$P_1\ (t_2 = \text{Flove} \mid t_1 = \text{Hlove}) = 1/3$
$P_2\ (t_1 = \text{Hlove} \mid t_2 = \text{Hlove}) = 1$
$P_2\ (t_1 = \text{Flove} \mid t_2 = \text{Hlove}) = 0$
$P_2\ (t_1 = \text{Hlove} \mid t_2 = \text{Flove}) = 1$
$P_2\ (t_1 = \text{Flove} \mid t_2 = \text{Hlove}) = 0$

$u_1\ (H,H,\text{Hlove}) = 2$ $u_2\ (H,H,\text{Hlove}) = 2$
$u_1\ (H,H,\text{Flove}) = 1$ $u_2\ (H,H,\text{Flove}) = 1$
$u_1\ (H,F,\text{Hlove}) = 0$ $u_2\ (H,F,\text{Hlove}) = 0$
$u_1\ (H,F,\text{Flove}) = 0$ $u_2\ (H,F,\text{Flove}) = 0$
$u_1\ (F,H,\text{Hlove}) = 0$ $u_2\ (F,H,\text{Hlove}) = 0$
$u_1\ (F,H,\text{Flove}) = 0$ $u_2\ (F,H,\text{Flove}) = 0$
$u_1\ (F,F,\text{Hlove}) = 1$ $u_2\ (F,F,\text{Hlove}) = 1$
$u_1\ (F,F,\text{Flove}) = 2$ $u_2\ (F,F,\text{Flove}) = 2$

Bayesian Nash Equilibrium

The set of strategies (s_1^* ,s_2^* $\cdots$ s_n^*) are a

Pure Strategy Bayesian Nash Equilibrium

iff for each player *i*, and for each possible type of *i* : $t_i \in T_i$

$s_i^*(t_i)$ =

$$\arg\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i\left(s_1^*(t_1),\ldots s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1})\ldots s_n^*(t_n)\right) \times \mathrm{P}_i\left(t_{-i} \middle| t_i\right)$$

i.e. no player, in any of their types, wants
to change their strategy

NEGOTIATION: A Bayesian Game

Two players:	S, (seller) and B, (buyer)
$T_s = [0,1]$	the seller's type is a real number between 0 and 1 specifying the value (in dollars) to them of the object they are selling
$T_b = [0,1]$	the buyer's type is also a real number. The value to the buyer.

Assume that at the start

$V_s \in T_s$ is chosen uniformly at random

$V_b \in T_b$ is chosen uniformly at random

The "Double Auction" Negotiation

S writes down a price for the item (g_s)

B simultaneously writes down a price (g_b)

Prices are revealed

If $g_s = g_b$ no trade occurs, both players have payoff 0

If $g_s = g_b$ then buyer pays the midpoint price $\frac{(g_s+g_b)}{2}$ and receives the item

Payoff to S : $1/2(g_s+g_b)-V_s$

Payoff to B : $V_b-1/2(g_s+g_b)$

Negotiation in Bayesian Game Notation

$T_s = [0,1]$ write $V_s \in T_s$
$T_b = [0,1]$ write $V_b \in T_b$

$P_s(V_b|V_s) = P_s(V_b)$ = uniform distribution on $[0,1]$
$P_b(V_s|V_b) = P_b(V_s)$ = uniform distribution on $[0,1]$

$A_s = [0,1]$ write $g_s \in A_s$
$A_b = [0,1]$ write $g_b \in A_b$

$u_s(P_s, P_b, V_s) =$ What?

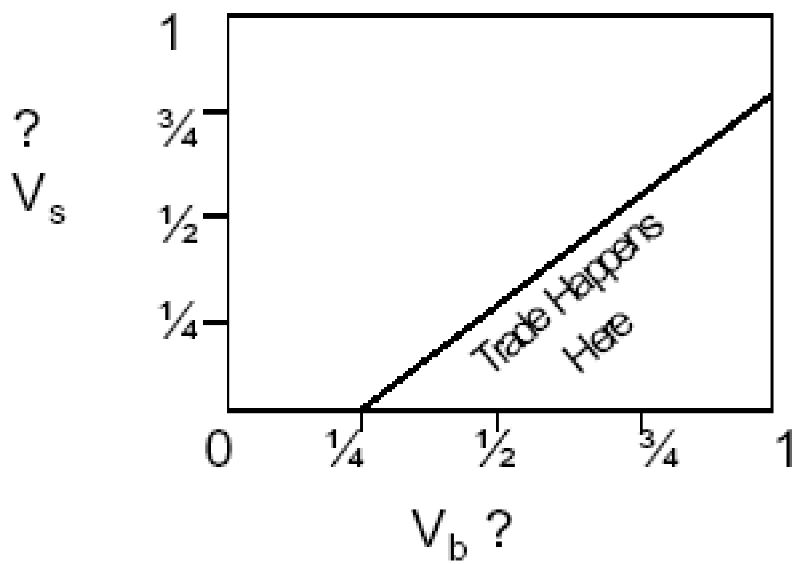
$u_b(P_s, P_b, V_b) =$ What?

Double Negotiation: When does trade occur?

…when

$$g_b^*(V_b) = 1/12 + 2/3\ V_b > 1/4 + 2/3\ V_s = g_s^*(V_s)$$

i.e. when $V_b > V_s + 1/4$



Prob(Trade Happens) = $1/2 \times (3/4)^2 = 9/32$

What You Should Know

Strict dominance

Nash Equilibria

Continuous games like Tragedy of the Commons

Rough, vague, appreciation of threats

Bayesian Game formulation

What You Shouldn't Know

- How many goats your lecturer has on his property
- What strategy Mephistopheles uses in his negotiations
- What strategy this University employs when setting tuition
- How to square a circle using only compass and straight edge
- How many of your friends and colleagues are active Santa informants, and how critical they've been of your obvious failings