Normative Decision Theory

A prescriptive theory for how decisions *should* be made to maximize the value of decision outcomes for an individual

Decision Theory

- Quantify preferences on outcomes *s*
  - U(*s*,*a*)
- Quantify Beliefs about outcomes of actions
  - $P(s|O,A)$ where
    - *O* are observations
    - *A* are actions
- Decision making principle:
  - Choose *A* that Maximizes Expected Utility
    - Needs link between s & A, $s' = T(s,A)$

Utility

Utility is:

- A numerical measure of goal achievement.
- A numerical measure of ``good''.

Utility isn't just:

- Pleasure.
- Money.
- Happiness.
- Outcomes that have ``usefulness'' for something else.

Utility Matrix

$$U(a,s)$$

OUTCOMES

ACTIONS

	Columns are different things		
Option	1	2	3
A	$u_A(1)$	$u_A(2)$	$u_A(3)$
B	$u_B(1)$	$u_B(2)$	$u_B(3)$

Can we boil all good down to a number?

- Probably not.
- Different kinds of utility (Kahneman):
  - Experienced utility
    - E.g. Pain during treatment
  - Remembered utility
    - E.g. Pain remembered after treatment
  - Predicted utility
    - *Do people know what will be good for them?*
  - Decision utility
    - *Do people use their knowledge when making decisions?*

Fundamental Equation

Value of a decision = Expected Utility of making an action A, where the expectation (average) is carried out over the possible outcomes of that action.

$$E\left[U(A\,|\,obs)\right] = \sum_{s} P(\mathbf{R}_{\mathbf{esult}\,s}(A)\,|\,obs, \mathbf{Do}(A))\,U(\mathbf{R}_{\mathbf{esult}\,s}(A))$$

$s = \mathbf{R}_{\mathbf{esult}\,s}(A)$ is the s^{th} possible outcome of action A

$$V = E\left[U[A\,|\,O]\right] = \sum_{s} \underbrace{P(s\,|\,O, A)}_{\text{Transition Matrix}}\ \underbrace{U(s, A)}_{\text{Utility Function}}$$

s : state of the world

O: observation

A : action

Preference Nomenclature

Lotteries:

A lottery is a probabilistic mixture of outcomes

$$L = [p, A;\ 1 - p, B]$$

Ordering using lotteries

$A \succ B$	A is preferred to B
$A \sim B$	the agent is indifferent between A and B
$A \succsim B$	the agent prefers A to B or is indifferent between them

Utility Theory Axioms 1

◇ **Orderability**: Given any two states, a rational agent must either prefer one to the other or else rate the two as equally preferable. That is, an agent should know what it wants.

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

◇ **Transitivity**: Given any three states, if an agent prefers A to B and prefers B to C, then the agent must prefer A to C.

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

◇ **Continuity**: If some state B is between A and C in preference, then there is some probability p for which the rational agent will be indifferent between getting B for sure and the lottery that yields A with probability p and C with probability $1 - p$.

$$A \succ B \succ C \Rightarrow \exists p \ [p, A;\ 1 - p, C] \sim B$$

Utility Axioms 2

- **Substitutability**: If an agent is indifferent between two lotteries, A and B, then the agent is indifferent between two more complex lotteries that are the same except that B is substituted for A in one of them. This holds regardless of the probabilities and the other outcome(s) in the lotteries.

$$A \sim B \Rightarrow [p, A;\ 1-p, C] \sim [p, B; 1-p, C]$$

- **Monotonicity**: Suppose there are two lotteries that have the same two outcomes, A and B. If an agent prefers A to B, then the agent must prefer the lottery that has a higher probability for A (and vice versa).

$$A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A;\ 1-p, B] \succsim [q, A;\ 1-q, B])$$

- **Decomposability**: Compound lotteries can be reduced to simpler ones using the laws of probability. This has been called the "no fun in gambling" rule because it says that an agent should not prefer (or disprefer) one lottery just because it has more choice points than another.[2]

$$[p, A;\ 1-p, [q, B;\ 1-q, C]] \sim [p, A;\ (1-p)q, B;\ (1-p)(1-q), C]$$

What do the Axioms do? They Guarantee:

1) Utility principle

$$U(A) > U(B) \Leftrightarrow A \succ B$$
$$U(A) = U(B) \Leftrightarrow A \sim B$$

There exists a monotonic function that numerically encodes preferences

2) Maximum expected utility principle

$$U([p_1, S_1; \ldots; p_n, S_n]) = \sum_i p_i U(S_i)$$

Utility of a lottery is the expection of the utilities

An Example: You bet your *what*?

You just won $1,000,000

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BUT

You are offered a gamble:

Bet your $1,000,000.00 on a fair coin flip.

Heads: $3,000,000

Tails: $0.00

What should you do?

Problem Analysis

Expected monetary gain = 0.5* \$0 + 0.5* \$ 3,000,000 = \$1,500,000

\$1,500,000 > \$ 1,000,000 !
Will you take the bet now?
How much do you need as a pay off?

Utility theory posits lotteries that result in indifference, and in taking the bet.

Let S_k be your current wealth. Let $U(S_k) = 5$; $U(S_{k+3,000,000}) = 10$; $U(S_{k+1,000,000}) = 8$;

$$EU(Accept) = \tfrac{1}{2}U(S_k) + \tfrac{1}{2}U(S_{k+3,000,000})$$

$$EU(Accept) = U(S_{k+1,000,000})$$

Bernoulli's Game

Given a fair coin

I will toss this coin N times until it comes up heads.

Your payoff = 2^N

Game Analysis

$$EMV(St.P.) = \sum_i P(Heads_i)MV(Heads_i)$$

$$= \sum_i \frac{1}{2^i}2^i = \frac{2}{2} + \frac{4}{4} + \frac{8}{8} + \cdots = \infty$$

You should be willing to bet any finite amount?

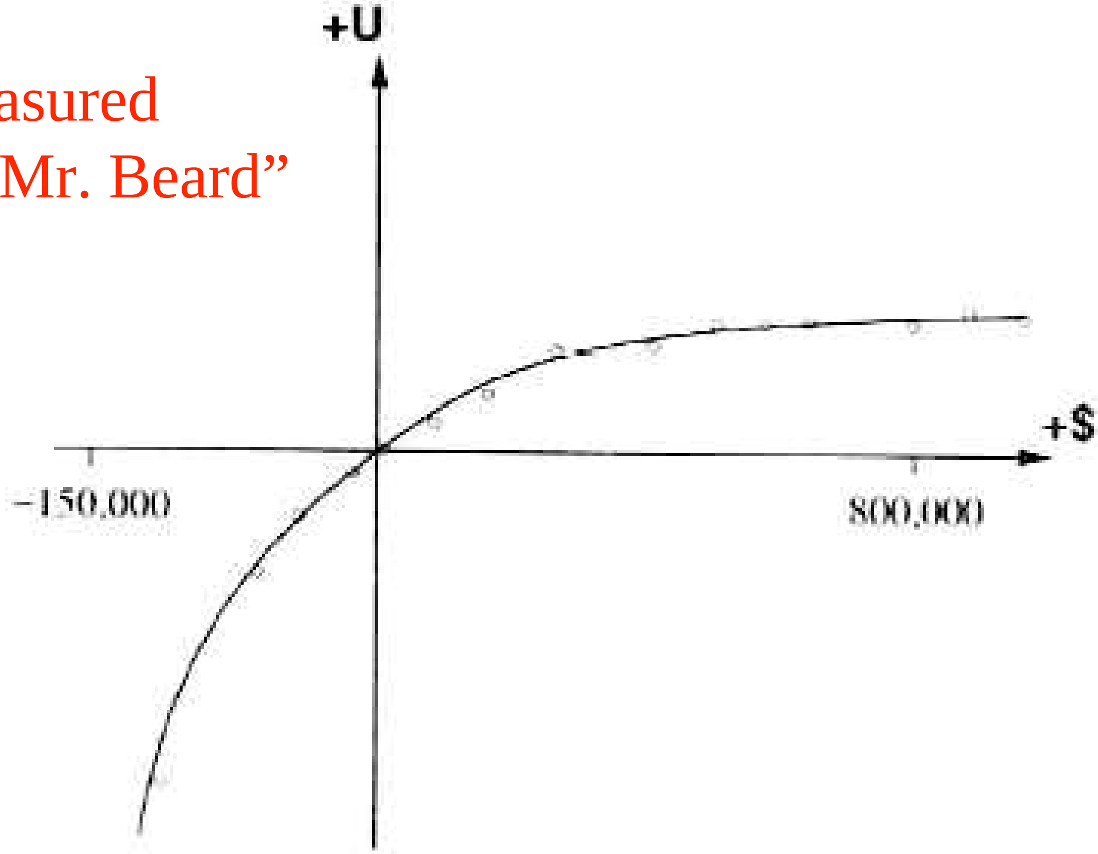
$$U(S_{k+n}) = \log_2 n \qquad (\text{for } n > 0)$$

$$EU(St.P.) = \sum_i P(Heads_i)U(Heads_i)$$

$$= \sum_i \frac{1}{2^i}\log_2 2^i = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots = 2$$

Measured Utility function

Utility function measured using lotteries for "Mr. Beard"
Grayson, 1960



$$U(S_{k+n}) = -263.31 + 22.09 \log(n + 150,000)$$

Some Violations

Game 1:

A: 80% chance winning $4000

B: 100% chance winning $3000

Result B>A

So $0.8\,U(x+\$4000) < U(x+\$3000)$

Game 2:

C: 20% chance winning $4000

D: 25% chance winning $3000

Result C preferred to D

So $(0.2/0.25)\,U(x+\$4000) > U(x+\$3000)$

$0.8\,U(x+\$4000) > U(x+\$3000)$

For people, preferences are sometimes a function of the probability

Another Violation

Lack of Independence of Irrelevant alternatives

Salmon $12.50

Steak $25.00

If restaurant is first-rate, Steak > Salmon

Restaurant looks kind of seedy => salmon

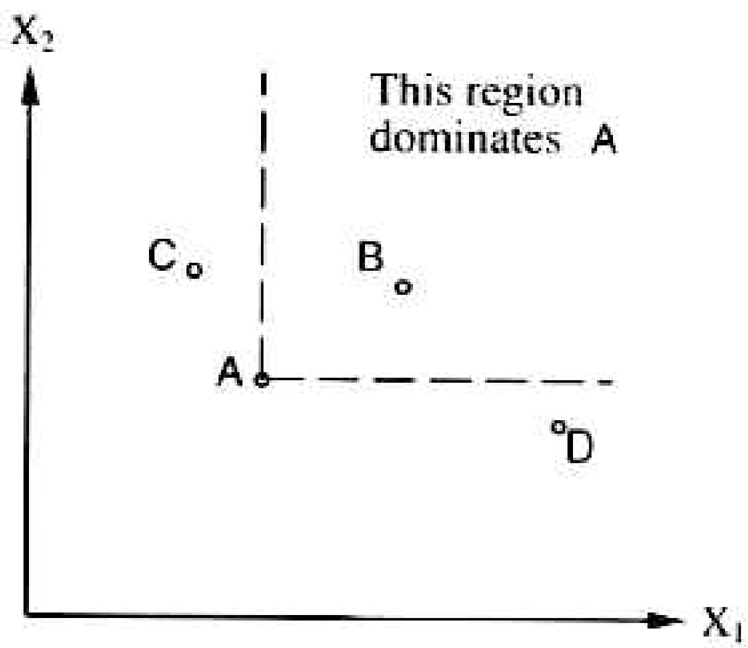
Waiter comes back and says he forgot to say they have snails and frog's legs

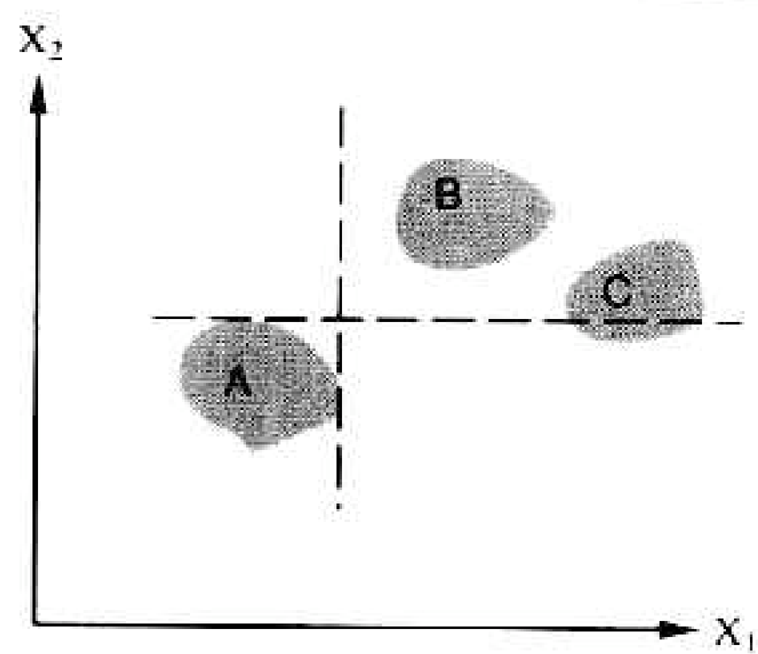
Man says "I'll have the steak"

Multi-attribute Utility

$$U(x_1, \ldots, x_n) = f[f_1(x_1), \ldots, f_n(x_n)]$$

Hopefully $f_i(x_i)$ are simply like addition





Utility functions

Perception:

- Utility measured by correctness of inference
- Utility measured by perceived energy expenditure

Action

- Utility end point accuracy
- Utility measured by minimum energy expenditure

Social

Utility functions for attractiveness?

But what's the use in beauty?

Money?

What else is there?

Rational Mate Choice?