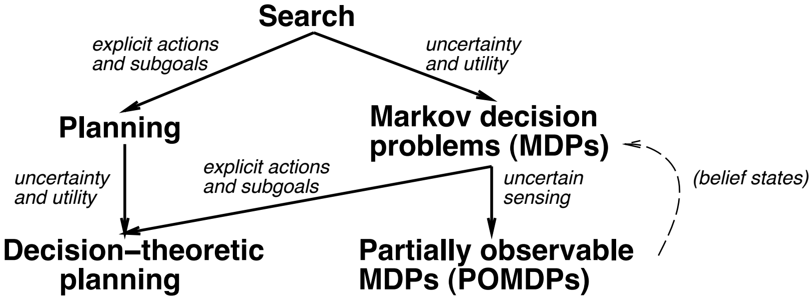
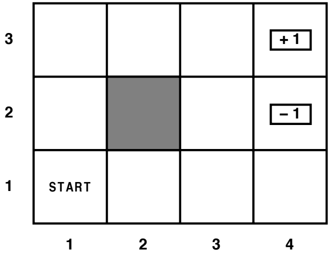
Making Complex Decisions

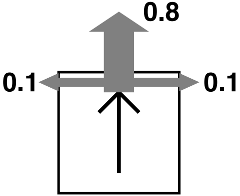
CSci 5512: Artificial Intelligence II

Sequential Decision Problems



Markov Decision Process



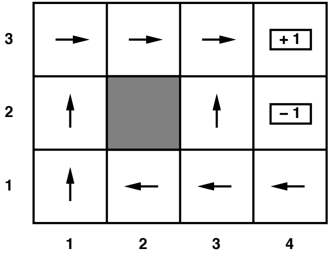


- States $s \in S$, actions $a \in A$
- Model $T(s, a, s') \equiv P(s'|s, a)$
- Reward function $R(s)$ (or $R(s, a)$, $R(s, a, s')$)

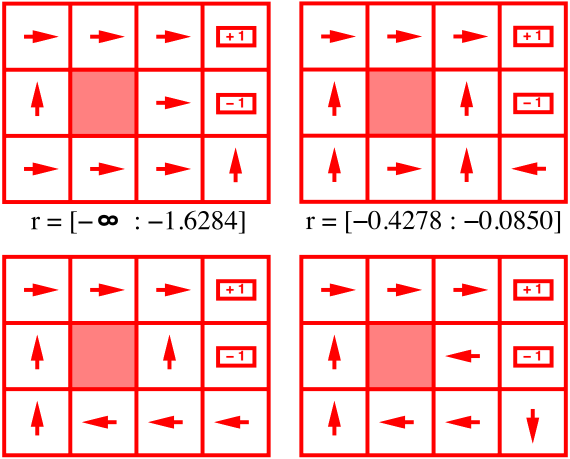
$$R(s) = \begin{cases} -0.04 & \text{(small penalty) for nonterminal states} \\ \pm 1 & \text{for terminal states} \end{cases}$$

Solving MDPs

- In search problems, aim is to find an optimal *sequence*
- In MDPs, aim is to find an optimal *policy* $\pi(s)$
  - Best action for every possible state s
  - Cannot predict where one will end up
- Optimal policy maximizes *expected sum of rewards*
- Optimal policy when state penalty $R(s)$ is –0.04:



Reward and Optimal Policy



r = [−∞ : −1.6284]

r = [−0.4278 : −0.0850]

r = [−0.0480 : −0.0274]

r = [−0.0218 : 0.0000]

Utility of State Sequences

- Need to understand preferences between *sequences* of states
- Typically consider stationary preferences on reward sequences

$$[r, r_0, r_1, r_2, \ldots] \succ [r, r'_0, r'_1, r'_2, \ldots] \Leftrightarrow [r_0, r_1, r_2, \ldots] \succ [r'_0, r'_1, r'_2, \ldots]$$

- <u>Theorem</u>: Only two ways to combine rewards over time:
  1) *Additive* utility function:

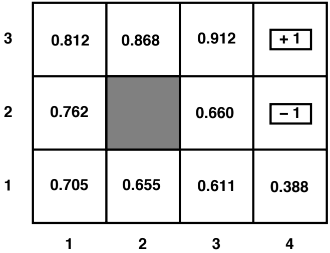
  $$U([s_0, s_1, s_2, \ldots]) = R(s_0) + R(s_1) + R(s_2) + \cdots$$

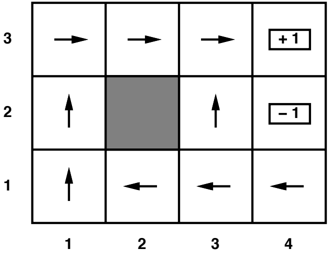
  2) *Discounted* utility function: For discount factor γ

  $$U([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots$$

Utility of States

- Utility $U(s)$ of a *state* (a.k.a. its *value*)
  - Expected (discounted) sum of rewards (until termination)
  - Assume optimal actions
- Choosing the best action is just MEU
  - Maximize the expected utility of the immediate successors
  - Utilities of the states are given





Utilities (Contd.)

Problem: Infinite lifetimes $\implies$ Additive utilities are infinite

- Finite horizon:
  - Termination at a *fixed time T*
  - Non-stationary policy as $\pi(s)$ depends on time left
- Absorbing state(s):
  - With probability 1, agent eventually "dies" for any π
  - Expected utility of every state is finite
- Discounting:
  - Assuming $\gamma < 1$, $R(s) \leq R_{\max}$,

$$U([s_0, \ldots s_\infty]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq R_{\max}/(1-\gamma)$$

- Average reward per time step
  - Theorem: $\pi^*(s)$ has constant gain after initial transient

The Optimal Policy

- Given a policy π, the overall (discounted) utility

$$U_\pi = \sum_{t=0}^{\infty} \gamma^t R(s_t)$$

- U_π a random variable as s_t are random
- Optimal policy corresponds to the MEU

$$\pi^* = \text{argmax}_\pi \, E[U_\pi] = \text{argmax}_\pi \, E\left[\sum_{t=0}^{\infty} \gamma^t R(s_t) | \pi\right]$$

Dynamic Programming: The Bellman Equation

- Simple relationship among utilities of neighboring states
- Expected sum of rewards $=$ current reward
  $+\ \gamma\times$ Expected sum of rewards after taking best action
- Bellman equation (1957):

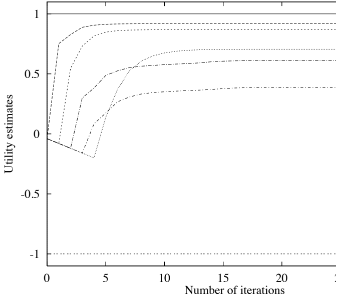
$$U(s) = R(s) + \gamma \max_a \sum_{s'} U(s')\, T(s,a,s')$$

- One equation per state s
  - n nonlinear equations in n unknowns

Value Iteration Algorithm

- Main Idea
  - Start with arbitrary utility values
  - Update to make them locally consistent
  - Everywhere locally consistent $\Rightarrow$ Global optimality
- Repeat for every s simultaneously until "no change"

$$U(s) \leftarrow R(s) + \gamma \max_a \sum_{s'} U(s')\,T(s,a,s') \qquad \forall s$$



Convergence of Value Iteration

- Define the max-norm $||U|| = \max_s |U(s)|$
  - $||U - V||$ = maximum difference between U and V
- Let U^t and U^{t+1} be successive approximations to the true U
- Theorem: For any two approximations U^t and V^t

$$||U^{t+1} - V^{t+1}|| \leq \gamma \, ||U^t - V^t||$$

- Any distinct approximations must get closer to each other
  - Any approximation must get closer to the true U
  - VI converges to a unique, stable, optimal solution
- Theorem: If $||U^{t+1} - U^t|| < \epsilon(1 - \gamma)/\gamma$, then $||U^{t+1} - U|| < \epsilon$
- Once the change in U^t becomes small, we are almost done
- MEU policy using U^t may be optimal long before convergence

Policy Iteration

- Search for optimal policy and utility values simultaneously
- Algorithm:
  1. $\pi \leftarrow$ an arbitrary initial policy
  2. Repeat until no change in π
     1. Compute utilities given π
     2. Update π as if utilities were correct (i.e., local MEU)

- To compute utilities given a fixed π (value determination):

$$U(s) = R(s) + \gamma \sum_{s'} U(s') T(s, \pi(s), s') \qquad \text{for all } s$$

- n linear equations in n unknowns, solve in $O(n^3)$

Modified Policy Iteration

- Policy iteration often converges in few iterations
  - But each iteration is expensive
- Main Idea:
  - An approximate value determination step
  - Use a few steps of value iteration (with π fixed)
  - Start from the value function produced in the last iteration
- Often converges much faster than pure VI or PI
- Leads to much more general algorithms
  - Value/Policy updates can be performed locally in any order
- Reinforcement learning algorithms use such updates

Partial Observability

- POMDP has an observation model
  - $O(s, e) = P(\text{obtain evidence } e \text{ when in state } s)$
- Agent does not know which state it is in
  - Makes no sense to talk about policy $\pi(s)$
- Theorem (Astrom, 1965): The optimal policy in a POMDP is a function $\pi(b)$ where b is the belief state (probability distribution over states)
- Can convert a POMDP into an MDP in belief-state space
- $T(b, a, b')$ is the probability that the new belief state is b' given that the current belief state is b and the agent does a
  - Essentially a filtering update step

Partial observability (Contd.)

- Solutions automatically include information-gathering behavior
- If there are n states, b is an n-dimensional real-valued vector
  - Solving POMDPs is very (actually, PSPACE-) hard
- The real world is a POMDP (with initially unknown T and O)