

# UNCERTAINTY

## CHAPTER 13

# Outline

- ◇ Uncertainty
- ◇ Probability
- ◇ Syntax and Semantics
- ◇ Inference
- ◇ Independence and Bayes' Rule

# Uncertainty

Let action  $A_t$  = leave for airport  $t$  minutes before flight  
Will  $A_t$  get me there on time?

Problems:

- 1) partial observability (road state, other drivers' plans, etc.)
- 2) noisy sensors (KCBS traffic reports)
- 3) uncertainty in action outcomes (flat tire, etc.)
- 4) immense complexity of modelling and predicting traffic

Hence a purely logical approach either

- 1) risks falsehood: " $A_{25}$  will get me there on time"
- or 2) leads to conclusions that are too weak for decision making:  
" $A_{25}$  will get me there on time if there's no accident on the bridge  
and it doesn't rain and my tires remain intact etc etc."

( $A_{1440}$  might reasonably be said to get me there on time  
but I'd have to stay overnight in the airport ...)

# Methods for handling uncertainty

Default or nonmonotonic logic:

Assume my car does not have a flat tire

Assume  $A_{25}$  works unless contradicted by evidence

Issues: What assumptions are reasonable? How to handle contradiction?

Rules with fudge factors:

$A_{25} \mapsto_{0.3} \textit{AtAirportOnTime}$

$\textit{Sprinkler} \mapsto_{0.99} \textit{WetGrass}$

$\textit{WetGrass} \mapsto_{0.7} \textit{Rain}$

Issues: Problems with combination, e.g., *Sprinkler* causes *Rain*??

Probability

Given the available evidence,

$A_{25}$  will get me there on time with probability 0.04

Mahaviracarya (9th C.), Cardano (1565) theory of gambling

(Fuzzy logic handles **degree of truth** NOT uncertainty e.g.,

*WetGrass* is true to degree 0.2)

# Probability

Probabilistic assertions **summarize** effects of

**laziness**: failure to enumerate exceptions, qualifications, etc.

**ignorance**: lack of relevant facts, initial conditions, etc.

Subjective or Bayesian probability:

Probabilities relate propositions to one's own state of knowledge

e.g.,  $P(A_{25} | \text{no reported accidents}) = 0.06$

These are **not** claims of a “probabilistic tendency” in the current situation  
(but might be learned from past experience of similar situations)

Probabilities of propositions change with new evidence:

e.g.,  $P(A_{25} | \text{no reported accidents, 5 a.m.}) = 0.15$

(Analogous to logical entailment status  $KB \models \alpha$ , not truth.)

# Making decisions under uncertainty

Suppose I believe the following:

$$P(A_{25} \text{ gets me there on time} | \dots) = 0.04$$

$$P(A_{90} \text{ gets me there on time} | \dots) = 0.70$$

$$P(A_{120} \text{ gets me there on time} | \dots) = 0.95$$

$$P(A_{1440} \text{ gets me there on time} | \dots) = 0.9999$$

Which action to choose?

Depends on my **preferences** for missing flight vs. airport cuisine, etc.

**Utility theory** is used to represent and infer preferences

**Decision theory** = utility theory + probability theory

# Probability basics

Begin with a set  $\Omega$ —the **sample space**

e.g., 6 possible rolls of a die.

$\omega \in \Omega$  is a **sample point/possible world/atomic event**

A **probability space** or **probability model** is a sample space with an assignment  $P(\omega)$  for every  $\omega \in \Omega$  s.t.

$$0 \leq P(\omega) \leq 1$$

$$\sum_{\omega} P(\omega) = 1$$

e.g.,  $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$ .

An **event**  $A$  is any subset of  $\Omega$

$$P(A) = \sum_{\{\omega \in A\}} P(\omega)$$

E.g.,  $P(\text{die roll} < 4) = P(1) + P(2) + P(3) = 1/6 + 1/6 + 1/6 = 1/2$

## Random variables

A **random variable** is a function from sample points to some range, e.g., the reals or Booleans

e.g.,  $Odd(1) = true$ .

$P$  induces a **probability distribution** for any r.v.  $X$ :

$$P(X = x_i) = \sum_{\{\omega: X(\omega) = x_i\}} P(\omega)$$

e.g.,  $P(Odd = true) = P(1) + P(3) + P(5) = 1/6 + 1/6 + 1/6 = 1/2$



# Propositions

Think of a proposition as the event (set of sample points) where the proposition is true

Given Boolean random variables  $A$  and  $B$ :

event  $a$  = set of sample points where  $A(\omega) = \text{true}$

event  $\neg a$  = set of sample points where  $A(\omega) = \text{false}$

event  $a \wedge b$  = points where  $A(\omega) = \text{true}$  and  $B(\omega) = \text{true}$

Often in AI applications, the sample points are **defined** by the values of a set of random variables, i.e., the sample space is the Cartesian product of the ranges of the variables

With Boolean variables, sample point = propositional logic model

e.g.,  $A = \text{true}$ ,  $B = \text{false}$ , or  $a \wedge \neg b$ .

Proposition = disjunction of atomic events in which it is true

e.g.,  $(a \vee b) \equiv (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)$

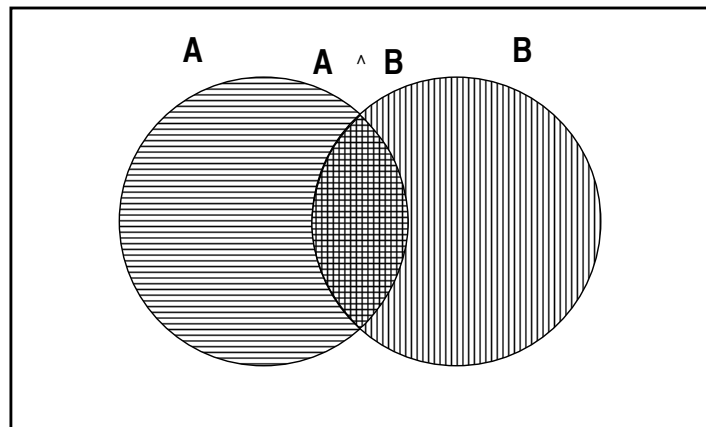
$\Rightarrow P(a \vee b) = P(\neg a \wedge b) + P(a \wedge \neg b) + P(a \wedge b)$

## Why use probability?

The definitions imply that certain logically related events must have related probabilities

E.g.,  $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$

True



de Finetti (1931): an agent who bets according to probabilities that violate these axioms can be forced to bet so as to lose money regardless of outcome.

# Syntax for propositions

Propositional or Boolean random variables

e.g., *Cavity* (do I have a cavity?)

*Cavity = true* is a proposition, also written *cavity*

Discrete random variables (*finite* or *infinite*)

e.g., *Weather* is one of  $\langle \textit{sunny}, \textit{rain}, \textit{cloudy}, \textit{snow} \rangle$

*Weather = rain* is a proposition

Values must be exhaustive and mutually exclusive

Continuous random variables (*bounded* or *unbounded*)

e.g., *Temp = 21.6*; also allow, e.g., *Temp < 22.0*.

Arbitrary Boolean combinations of basic propositions

## Prior probability

Prior or unconditional probabilities of propositions

e.g.,  $P(\text{Cavity} = \text{true}) = 0.1$  and  $P(\text{Weather} = \text{sunny}) = 0.72$   
correspond to belief prior to arrival of any (new) evidence

Probability distribution gives values for all possible assignments:

$$\mathbf{P}(\text{Weather}) = \langle 0.72, 0.1, 0.08, 0.1 \rangle \text{ (normalized, i.e., sums to 1)}$$

Joint probability distribution for a set of r.v.s gives the probability of every atomic event on those r.v.s (i.e., every sample point)

$\mathbf{P}(\text{Weather}, \text{Cavity})$  = a  $4 \times 2$  matrix of values:

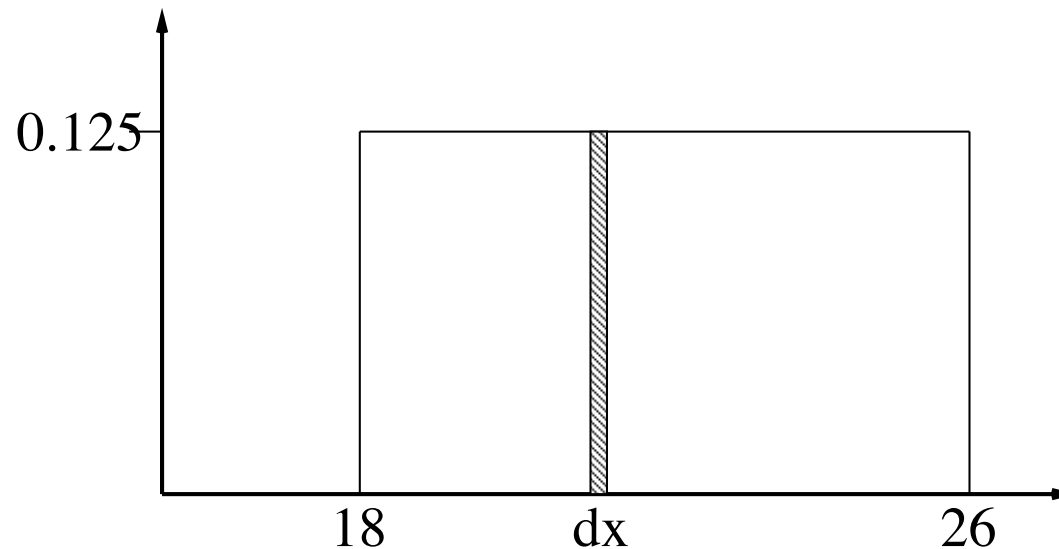
<i>Weather</i> =	<i>sunny</i>	<i>rain</i>	<i>cloudy</i>	<i>snow</i>
<i>Cavity</i> = <i>true</i>	0.144	0.02	0.016	0.02
<i>Cavity</i> = <i>false</i>	0.576	0.08	0.064	0.08

Every question about a domain can be answered by the joint distribution because every event is a sum of sample points

# Probability for continuous variables

Express distribution as a parameterized function of value:

$$P(X = x) = U[18, 26](x) = \text{uniform density between } 18 \text{ and } 26$$



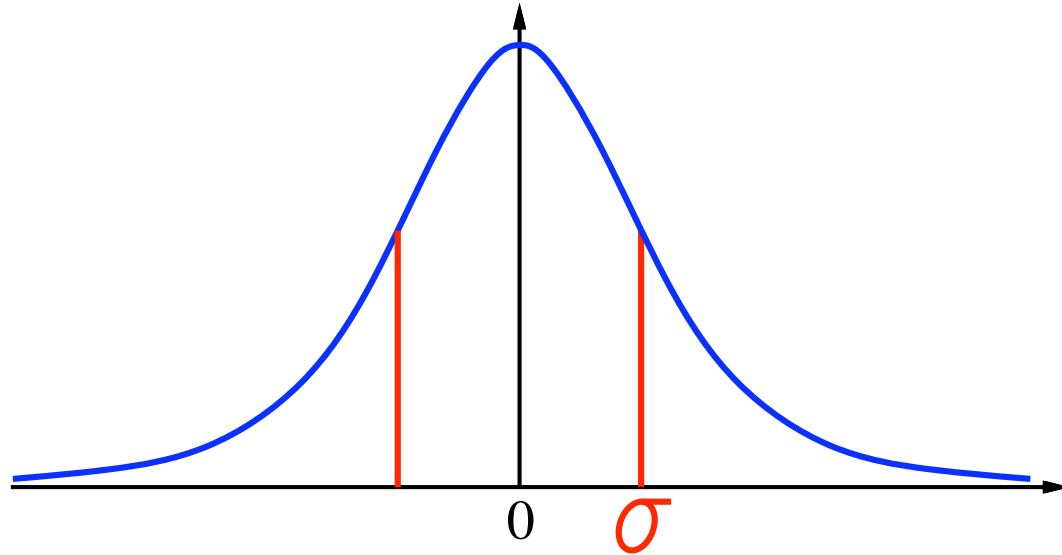
Here  $P$  is a **density**; integrates to 1.

$P(X = 20.5) = 0.125$  really means

$$\lim_{dx \rightarrow 0} P(20.5 \leq X \leq 20.5 + dx) / dx = 0.125$$

# Gaussian density

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



## Conditional probability

Conditional or posterior probabilities

e.g.,  $P(\text{cavity}|\text{toothache}) = 0.8$

i.e., **given that toothache is all I know**

**NOT** “if *toothache* then 80% chance of *cavity*”

(Notation for conditional distributions:

$P(\text{Cavity}|\text{Toothache}) = 2\text{-element vector of } 2\text{-element vectors})$

If we know more, e.g., *cavity* is also given, then we have

$P(\text{cavity}|\text{toothache}, \text{cavity}) = 1$

Note: the less specific belief **remains valid** after more evidence arrives, but is not always **useful**

New evidence may be irrelevant, allowing simplification, e.g.,

$P(\text{cavity}|\text{toothache}, \text{49ersWin}) = P(\text{cavity}|\text{toothache}) = 0.8$

This kind of inference, sanctioned by domain knowledge, is crucial

# Conditional probability

Definition of conditional probability:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

Product rule gives an alternative formulation:

$$P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$$

A general version holds for whole distributions, e.g.,

$$\mathbf{P}(Weather, Cavity) = \mathbf{P}(Weather|Cavity)\mathbf{P}(Cavity)$$

(View as a  $4 \times 2$  set of equations, **not** matrix mult.)

Chain rule is derived by successive application of product rule:

$$\begin{aligned} \mathbf{P}(X_1, \dots, X_n) &= \mathbf{P}(X_1, \dots, X_{n-1}) \mathbf{P}(X_n|X_1, \dots, X_{n-1}) \\ &= \mathbf{P}(X_1, \dots, X_{n-2}) \mathbf{P}(X_{n-1}|X_1, \dots, X_{n-2}) \mathbf{P}(X_n|X_1, \dots, X_{n-1}) \\ &= \dots \\ &= \prod_{i=1}^n \mathbf{P}(X_i|X_1, \dots, X_{i-1}) \end{aligned}$$



## Inference by enumeration

Start with the joint distribution:

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

For any proposition  $\phi$ , sum the atomic events where it is true:

$$P(\phi) = \sum_{\omega: \omega \models \phi} P(\omega)$$

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$$P(\text{toothache}) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$$

## Inference by enumeration

Start with the joint distribution:

	<i>toothache</i>		$\neg$ <i>toothache</i>	
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For any proposition  $\phi$ , sum the atomic events where it is true:

$$P(\phi) = \sum_{\omega: \omega \models \phi} P(\omega)$$

$$P(\text{cavity} \vee \text{toothache}) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$$

## Inference by enumeration

Start with the joint distribution:

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
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Can also compute conditional probabilities:

$$\begin{aligned}
 P(\neg \text{cavity} | \text{toothache}) &= \frac{P(\neg \text{cavity} \wedge \text{toothache})}{P(\text{toothache})} \\
 &= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4
 \end{aligned}$$

# Normalization

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	.072	.008
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	.144	.576

Denominator can be viewed as a normalization constant  $\alpha$

$$\begin{aligned}
 \mathbf{P}(Cavity|toothache) &= \alpha \mathbf{P}(Cavity, toothache) \\
 &= \alpha [\mathbf{P}(Cavity, toothache, catch) + \mathbf{P}(Cavity, toothache, \neg catch)] \\
 &= \alpha [\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle] \\
 &= \alpha \langle 0.12, 0.08 \rangle = \langle 0.6, 0.4 \rangle
 \end{aligned}$$

General idea: compute distribution on query variable  
by fixing **evidence variables** and summing over **hidden variables**

## Inference by enumeration, contd.

Let  $\mathbf{X}$  be all the variables. Typically, we want the posterior joint distribution of the query variables  $\mathbf{Y}$  given specific values  $\mathbf{e}$  for the evidence variables  $\mathbf{E}$

Let the hidden variables be  $\mathbf{H} = \mathbf{X} - \mathbf{Y} - \mathbf{E}$

Then the required summation of joint entries is done by summing out the hidden variables:

$$\mathbf{P}(\mathbf{Y}|\mathbf{E} = \mathbf{e}) = \alpha \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h}} \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h})$$

The terms in the summation are joint entries because  $\mathbf{Y}$ ,  $\mathbf{E}$ , and  $\mathbf{H}$  together exhaust the set of random variables

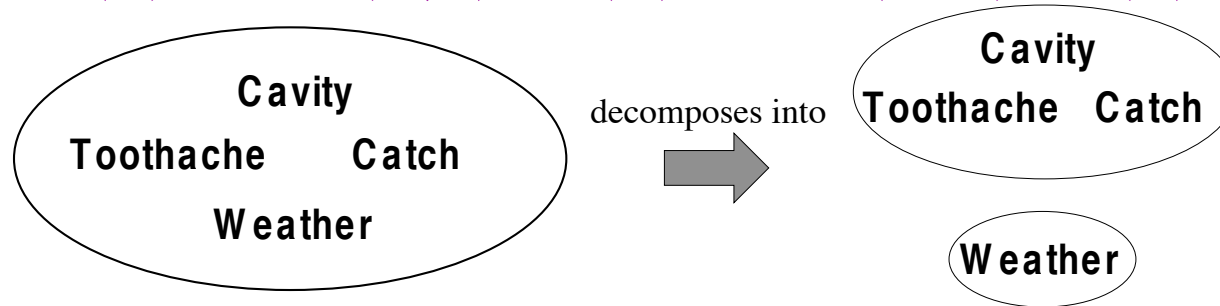
Obvious problems:

- 1) Worst-case time complexity  $O(d^n)$  where  $d$  is the largest arity
- 2) Space complexity  $O(d^n)$  to store the joint distribution
- 3) How to find the numbers for  $O(d^n)$  entries???

# Independence

$A$  and  $B$  are independent iff

$$\mathbf{P}(A|B) = \mathbf{P}(A) \quad \text{or} \quad \mathbf{P}(B|A) = \mathbf{P}(B) \quad \text{or} \quad \mathbf{P}(A, B) = \mathbf{P}(A)\mathbf{P}(B)$$



$$\begin{aligned} &\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity}, \textit{Weather}) \\ &= \mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})\mathbf{P}(\textit{Weather}) \end{aligned}$$

32 entries reduced to 12; for  $n$  independent biased coins,  $2^n \rightarrow n$

Absolute independence powerful but rare

Dentistry is a large field with hundreds of variables,  
none of which are independent. What to do?

## Conditional independence

$\mathbf{P}(\textit{Toothache}, \textit{Cavity}, \textit{Catch})$  has  $2^3 - 1 = 7$  independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

$$(1) \ P(\textit{catch}|\textit{toothache}, \textit{cavity}) = P(\textit{catch}|\textit{cavity})$$

The same independence holds if I haven't got a cavity:

$$(2) \ P(\textit{catch}|\textit{toothache}, \neg \textit{cavity}) = P(\textit{catch}|\neg \textit{cavity})$$

*Catch* is **conditionally independent** of *Toothache* given *Cavity*:

$$\mathbf{P}(\textit{Catch}|\textit{Toothache}, \textit{Cavity}) = \mathbf{P}(\textit{Catch}|\textit{Cavity})$$

Equivalent statements:

$$\mathbf{P}(\textit{Toothache}|\textit{Catch}, \textit{Cavity}) = \mathbf{P}(\textit{Toothache}|\textit{Cavity})$$

$$\mathbf{P}(\textit{Toothache}, \textit{Catch}|\textit{Cavity}) = \mathbf{P}(\textit{Toothache}|\textit{Cavity})\mathbf{P}(\textit{Catch}|\textit{Cavity})$$



## Conditional independence contd.

Write out full joint distribution using chain rule:

$$\begin{aligned} & \mathbf{P}(Toothache, Catch, Cavity) \\ &= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch, Cavity) \\ &= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity) \\ &= \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity) \end{aligned}$$

I.e.,  $2 + 2 + 1 = 5$  independent numbers (equations 1 and 2 remove 2)

In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in  $n$  to linear in  $n$ .

**Conditional independence is our most basic and robust form of knowledge about uncertain environments.**

## Bayes' Rule

Product rule  $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$$\Rightarrow \text{Bayes' rule } P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha \mathbf{P}(X|Y)\mathbf{P}(Y)$$

Useful for assessing **diagnostic** probability from **causal** probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

E.g., let  $M$  be meningitis,  $S$  be stiff neck:

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

Note: posterior probability of meningitis still very small!

# Bayes' Rule and conditional independence

$$\begin{aligned} & \mathbf{P}(Cavity|toothache \wedge catch) \\ &= \alpha \mathbf{P}(toothache \wedge catch|Cavity)\mathbf{P}(Cavity) \\ &= \alpha \mathbf{P}(toothache|Cavity)\mathbf{P}(catch|Cavity)\mathbf{P}(Cavity) \end{aligned}$$

This is an example of a **naive Bayes** model:

$$\mathbf{P}(Cause, Effect_1, \dots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i|Cause)$$



Total number of parameters is **linear** in  $n$

# Wumpus World

1,4	2,4	3,4	4,4
1,3	2,3	3,3	4,3
1,2 B OK	2,2	3,2	4,2
1,1 OK	2,1 B OK	3,1	4,1

$P_{ij} = \text{true}$  iff  $[i, j]$  contains a pit

$B_{ij} = \text{true}$  iff  $[i, j]$  is breezy

Include only  $B_{1,1}$ ,  $B_{1,2}$ ,  $B_{2,1}$  in the probability model

## Specifying the probability model

The full joint distribution is  $\mathbf{P}(P_{1,1}, \dots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1})$

Apply product rule:  $\mathbf{P}(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \dots, P_{4,4})\mathbf{P}(P_{1,1}, \dots, P_{4,4})$

(Do it this way to get  $P(\textit{Effect} \mid \textit{Cause})$ .)

First term: 1 if pits are adjacent to breezes, 0 otherwise

Second term: pits are placed randomly, probability 0.2 per square:

$$\mathbf{P}(P_{1,1}, \dots, P_{4,4}) = \prod_{i,j=1,1}^{4,4} \mathbf{P}(P_{i,j}) = 0.2^n \times 0.8^{16-n}$$

for  $n$  pits.

## Observations and query

We know the following facts:

$$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$$

$$known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$$

Query is  $\mathbf{P}(P_{1,3}|known, b)$

Define  $Unknown = P_{ij}$ s other than  $P_{1,3}$  and  $Known$

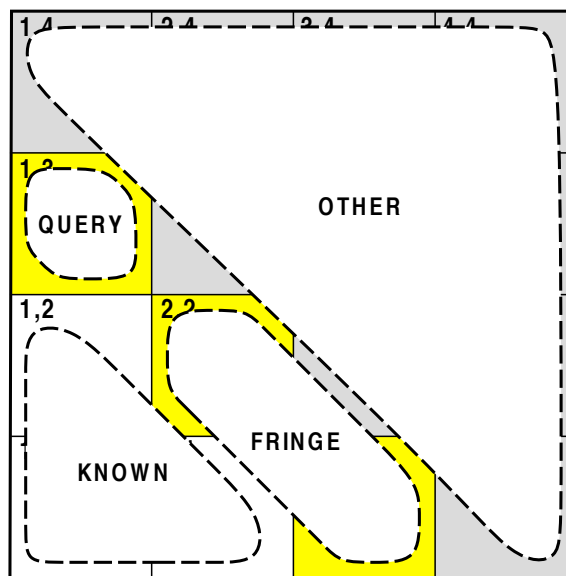
For inference by enumeration, we have

$$\mathbf{P}(P_{1,3}|known, b) = \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b)$$

Grows exponentially with number of squares!

## Using conditional independence

Basic insight: observations are conditionally independent of other hidden squares given neighbouring hidden squares



Define  $Unknown = Fringe \cup Other$

$$\mathbf{P}(b|P_{1,3}, Known, Unknown) = \mathbf{P}(b|P_{1,3}, Known, Fringe)$$

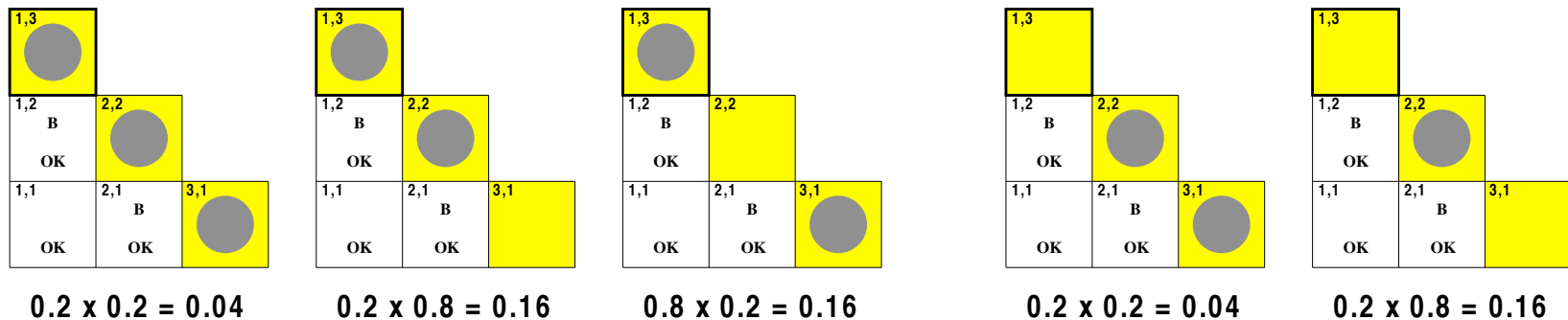
Manipulate query into a form where we can use this!

## Using conditional independence contd.

$$\begin{aligned}
 \mathbf{P}(P_{1,3} | \text{known}, b) &= \alpha \sum_{\text{unknown}} \mathbf{P}(P_{1,3}, \text{unknown}, \text{known}, b) \\
 &= \alpha \sum_{\text{unknown}} \mathbf{P}(b | P_{1,3}, \text{known}, \text{unknown}) \mathbf{P}(P_{1,3}, \text{known}, \text{unknown}) \\
 &= \alpha \sum_{\text{fringe}} \sum_{\text{other}} \mathbf{P}(b | \text{known}, P_{1,3}, \text{fringe}, \text{other}) \mathbf{P}(P_{1,3}, \text{known}, \text{fringe}, \text{other}) \\
 &= \alpha \sum_{\text{fringe}} \sum_{\text{other}} \mathbf{P}(b | \text{known}, P_{1,3}, \text{fringe}) \mathbf{P}(P_{1,3}, \text{known}, \text{fringe}, \text{other}) \\
 &= \alpha \sum_{\text{fringe}} \mathbf{P}(b | \text{known}, P_{1,3}, \text{fringe}) \sum_{\text{other}} \mathbf{P}(P_{1,3}, \text{known}, \text{fringe}, \text{other}) \\
 &= \alpha \sum_{\text{fringe}} \mathbf{P}(b | \text{known}, P_{1,3}, \text{fringe}) \sum_{\text{other}} \mathbf{P}(P_{1,3}) P(\text{known}) P(\text{fringe}) P(\text{other}) \\
 &= \alpha P(\text{known}) \mathbf{P}(P_{1,3}) \sum_{\text{fringe}} \mathbf{P}(b | \text{known}, P_{1,3}, \text{fringe}) P(\text{fringe}) \sum_{\text{other}} P(\text{other}) \\
 &= \alpha' \mathbf{P}(P_{1,3}) \sum_{\text{fringe}} \mathbf{P}(b | \text{known}, P_{1,3}, \text{fringe}) P(\text{fringe})
 \end{aligned}$$



# Using conditional independence contd.



$$\mathbf{P}(P_{1,3}|known, b) = \alpha' \langle 0.2(0.04 + 0.16 + 0.16), 0.8(0.04 + 0.16) \rangle \\ \approx \langle 0.31, 0.69 \rangle$$

$$\mathbf{P}(P_{2,2}|known, b) \approx \langle 0.86, 0.14 \rangle$$

## Summary

Probability is a rigorous formalism for uncertain knowledge

Joint probability distribution specifies probability of every atomic event

Queries can be answered by summing over atomic events

For nontrivial domains, we must find a way to reduce the joint size

Independence and conditional independence provide the tools