Example: Least Square Line Fitting

Data scatter

Data as 2D vectors

\[ a x_i + b y_i = c \]

\[ a \cdot x = c = \|x\| \cos \phi \]
Introducing Homogenous Coordinates

Data as 3D homogenous vectors \( p_i = [x_i \ y_i \ 1]' \)

In 3D, the set of points lies Close to a common plane

\[
a x_i + b y_i = c
\]

\[
a \cdot x = c = ||x|| \cos \phi
\]

Becomes

\[
a x_i + b y_i + c \ 1 = 0
\]

\[
a \cdot p_i = 0
\]
Geometry of solution
\[ A = \begin{pmatrix} -0.5 & -0.5 & 0.5 & 0.5 \\ 0.25 & -0.25 & -0.25 & 0.25 \\ 2 & 2 & 2 & 2 \end{pmatrix} \]

\[ [U, S, V] = \text{svd}(A) \]

\[ U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \]

\[ S = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{pmatrix} \]
Combining the Euclidean transformation and the scaling factor gives us a similarity transform.

- Combining the Euclidean transformation and the scaling factor gives us a similarity transform.
- Can only determine the location of the road up to a Euclidean transformation from the world coordinate frame.
- Eg. from image points alone we cannot determine latitude and longitude or which way is north.
- Cannot determine scale from image points either.
- Hence the scaling factor.

\[
T = \begin{bmatrix}
0 & \frac{1}{\kappa} \\
\kappa & 0
\end{bmatrix}
\]

rotation
translation
scaling by \(\kappa^{-1}\)
Fundamental Matrix, why are 8 point matches enough?

\[
\begin{bmatrix}
    u' \\
    v' \\
    1
\end{bmatrix}
= \begin{bmatrix}
    f_{11} & f_{12} & f_{13} \\
    f_{21} & f_{22} & f_{23} \\
    f_{31} & f_{32} & f_{33}
\end{bmatrix}
\begin{bmatrix}
    u \\
    v \\
    1
\end{bmatrix}
\]

\[f_{33} = 1\]

Thus only 8 free parameters =>
Need 8 or more constraints.
Stereo Reconstruction Ambiguity

- Without knowledge of scene's placement with respect to a 3D coordinate frame it is not possible to determine the absolute position and orientation of the scene from 2 (or any number of) views.
• What does this mean mathematically?
• Given
  – a set of 3D points $\tilde{X}_i$
  – two cameras $P, P'$ and
  – image points $\tilde{x}_i, \tilde{x}'_i$

• Remember these are related by:

$$\tilde{x}_i = P\tilde{X}_i$$
$$\tilde{x}'_i = P'\tilde{X}_i$$
• Replacing 
  \( \tilde{X}_t \) with \( T_{sim} \tilde{X}_t \),
  
  \( P, P' \) with \( PT_{sim}^{-1}, P'T_{sim}^{-1} \)

• Does not change the observed image points

  \[ \tilde{x}_t = PX_t \]
  \[ = (PT_{sim}^{-1})(T_{sim} \tilde{X}_t) \]
  \[ \tilde{x'}_t = P'X'_t \]
  \[ = (P'T_{sim}^{-1})(T_{sim} \tilde{X'}_t) \]
Extrinsic Parameter ambiguity

• If the camera calibration matrices are not known then the scene can only be constructed up to a *projective transformation* of the actual structure.

• A projective transformation is a homogeneous transformation of the form

\[ \tilde{X}_{\text{new}} = T_{\text{proj}} \tilde{X} \]

• where $T_{\text{proj}}$ is any 4×4 invertible matrix.
Projective structure

- What does a projective transformation look like?
- There are a whole family of these warped structures.
- Projective transformations
  - Map lines to lines
  - Preserve intersection and tangency if surfaces in contact.

2 views of a structure projectively equivalent to the true structure
Metric Structure

- If control points are available we can go from a projective to a true metric reconstruction.
- A projective reconstruction can be upgraded to a true metric reconstruction by specifying the 3D locations of 5 (or more) world points.
\[
B = 
\begin{bmatrix}
1 & -0.5 & -0.5 \\
0 & 0.86603 & -0.86603 \\
2 & 2 & 2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & -1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
[U,S,V] = \text{svd}(B)
\]

\[
U = 
\begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & -1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
[V,D] = \text{eig}(B)
\]

Yields z-axis and Complex eigenvalues
Representing the ambiguity
Matlab examples
Homework #2 solns
Problem #2
Projected image of a cube
Problem #3

3) Show that a straight line in 3D projects to a straight line in an image, using the simplest projection model, \( u = f \frac{x}{z} \), \( v = f \frac{y}{z} \).

Hint: A straight line can be defined by two points \( L = a*p1+(1-a)*p2 \), where \( a \) is a scalar.

\( p1 = [x1,y1,z1]' \);

Show that a third point on the line \( p3 = a*p1+(1-a)*p2 \) projects to an image point \( (u3,v3) \) that can be written in the form

\( u3 = b \ u1 + (1-b) \ u2 \)
\( v3 = b \ v1 + (1-b) \ v2 \)

where \( b = a*z1/(a \ z1 + (1-a) \ z2) \)
\[ p_1 = \frac{1}{z_1} M x_1 \]
\[ p_2 = \frac{1}{z_2} M x_2 \]
\[ x_3 = a \ x_1 \ +(1-a) \ x_2 \]

\[ p_3 = \frac{1}{z_3} M x_3 \]
\[ p_3 = a \ \frac{1}{z_3} M x_1 \ +(1-a) \ \frac{1}{z_3} M x_2 \]
\[ p_3 = a \ \frac{z_1}{z_3} p_1 \ +(1-a) \ \frac{z_2}{z_3} p_2 \]

But now \[ x_3 = a \ x_1 \ +(1-a) \ x_2 \]
And \[ z_3 = a \ z_1 \ +(1-a) \ z_2 \]
Thus
\[ p_3 = a \ \frac{z_1}{(a \ z_1 \ +(1-a) \ z_2)} \ p_1 \ +(1-a) \ \frac{z_2}{(a \ z_1 \ +(1-a) \ z_2)} \ p_2 \]

Set \[ b = a \ \frac{z_1}{(a \ z_1 \ +(1-a) \ z_2)} \]
Then \[ 1-b = (1-a) \ \frac{z_2}{(a \ z_1 \ +(1-a) \ z_2)} \]
And we have:
\[ p_3 = b \ p_1 \ +(1-b) \ p_2 \]  \{QED\}