Computational Vision
U. Minn. Psy 5036
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Lecture 5

# Goals

## Last time

### Ideal Observer Analysis: Essential idea

**Ideal observer**

Model the data (image) generation process

Define the inference task

Determine optimal performance

**Compare human performance to the ideal**

Ideal normalizes for information available

**Explain discrepancies in terms of:**

functional adaptation

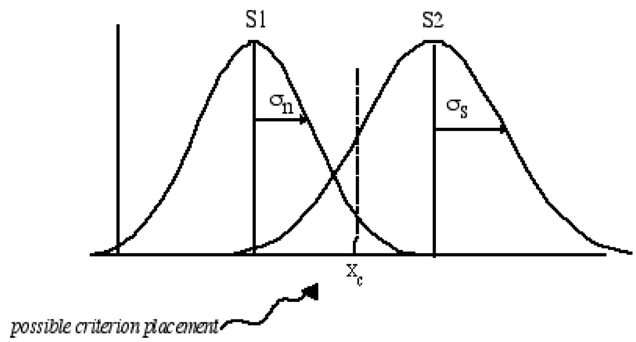
mechanism

## Psychophysical tasks & techniques (from the previous lecture)

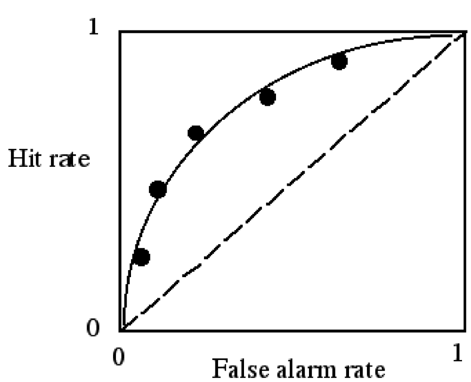
## The Receiver Operating Characteristic (ROC)

Although we can't directly measure the internal distributions of a human observer's decision variable, we've seen that we can measure hit and false alarm rates, and thus d'.

But one can do more, and use ROC measurements to see if an observer's decisions are consistent with Gaussian distributions with equal variance. If the criterion is varied, we can obtain a set of n data point. To get meaningful and equal d's for each pair of hit and false alarm rates assumes that the underlying relative separation of the signal and noise distributions remain unchanged and that the distributions are Gaussian, with equal standard deviation. We might know this is true (or true to a good approximation) for the ideal, but we have no guarantee for the human observer. Is there a way to check? Suppose the signal and noise distributions look like:




If we plot the hit rate vs. false alarm rate data on a graph as the criterion xc varies, we get something that looks like:




One can show that the area under the ROC curve is equal to the proportion correct in a two-alternative forced-choice experiment (Green and Swets).

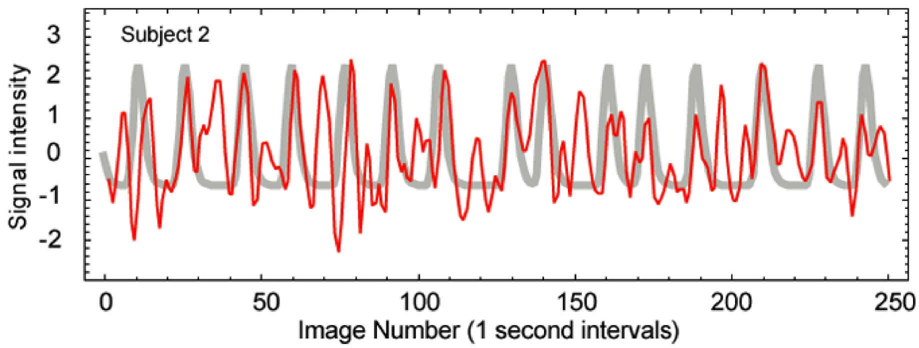
Sometimes, *sensitivity is operationally defined as this area*. This provides a single summary number, even if the standard definition of d' is inappropriate, for example because the variances are not equal.

We also saw that one can test the gaussian equal variance assumption by re-plotting the ROC curve in terms of the z-scores of the hit and false alarm rates.

## Applications of ROC to neural measures

The area under the ROC curve provides a useful measure of sensitivity even if the additive gaussian model isn't known to be correct. It can also be thought of as a measure of how much information about signal vs. no signal can be extracted from the data. ROC curves can be used characterize the sensitivity of single neurons, as well as gross overall measures of activity such as comes from brain imaging data.

In the figure below, the gray lines represent a behavioral response by a human observer--i.e. when the signal is high, the observer is indicating subjective "detection". The red lines represent a measured brain signal. How well does the brain signal predict what the observer is reporting?




## The 2AFC (two-alternative forced-choice) method

Usually rather than manipulating the criterion, we would rather do the experiment in such a way that it does not change. Is there a way to reduce the of a fluctuating criterion?

- **See the "mini-experiment" from "Statistical efficiency: competing with the ideal observer in a 2AFC task" in the last lecture**
- **Relating performance (proportion correct) to signal-to-noise ratio, d'.**

In psychophysics, the most common way to minimize the problem of a varying criterion is to use a two-alternative forced-choice procedure (2AFC). In a 2AFC task the observer is presented on each trial a pair of stimuli. One stimulus has the signal (e.g. high flash), and the other the noise (e.g. low flash). The order, however, is randomized. So if they are presented temporally, the signal or the noise might come first, but the observer doesn't know which from trial to trial. In the spatial version, the signal could be on the left of the computer screen with the noise on the right, or vice versa. One can show that for 2AFC:

$$d' = -\sqrt{2}\ z\,(\text{proportion correct}) \tag{1}$$

■ **Calculating the Pattern Ideal's d' for a two-alternative forced-choice experiment from a z-score of the proportion correct. (see Homework Assignment #1)**

For our 2AFC experiment, the observer gets two images to compare. One has the signal plus noise, and the other just noise. But the observer doesn't know which one is which. This strategy will result in a single measureable number, the proportion correct, Pc.

d' for a 2AFC task is given by the formula:

$$d' = -\sqrt{2} Z(P_c)$$

$$P = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-x^2/2} dx$$

### Today

Review some probability and statistics

Pattern detection: The signal-known-exactly (SKE) ideal observer

Demo of 2AFC for pattern detection in noise

# What does the eye see best?

### Make the question precise by asking:

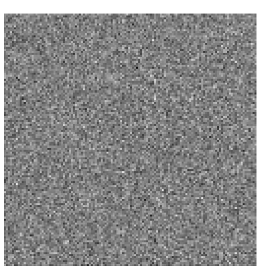
*For what patterns does the human visual system have the highest detection efficiencies relative to an ideal observer?*

- **Animals, or particularly dangerous ones?**

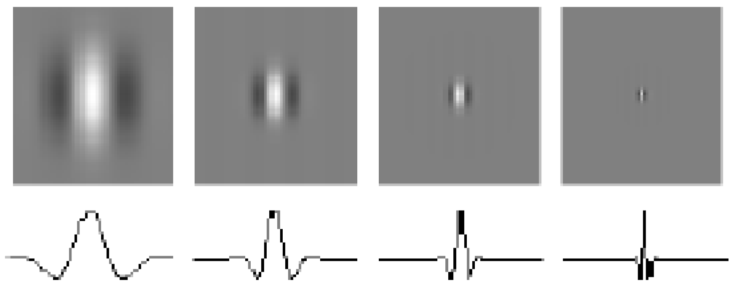
- **Faces, or a particular face?**

- **Or something simple, like a spot?**

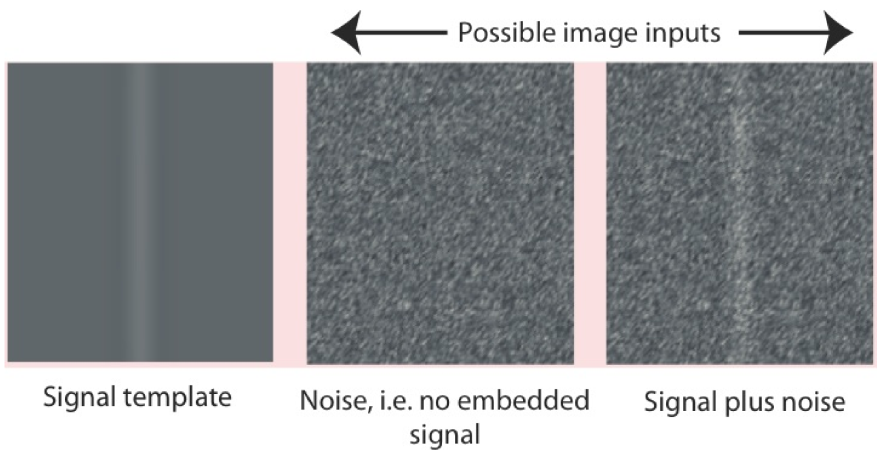
- **Or something complex, like a "frozen" noise image?**



**Or some pattern motivated by neurophysiology? E.g. the kinds of spatial patterns preferred by single neurons in the primary visual cortex ...**



Answering this question requires one to first devise a generative model that describes the variations in both the signal and the non-signal conditions. In general this is hard to do, but we can do it for simple cases such as when the signal image is constant, and the data is either "white gaussian noise" or the signal added to white gaussian noise.




## ■ Some intuition: Measures of pattern similarity

The fundamental problem of pattern recognition is deciding whether an input pattern **x** matches a stored representation **s**. This decision requires some measure of comparison between the input and the stored "template" **s**.

Given two patterns represented by vectors **x** and **s**, how can we measure how close or similar they are?

Some possibilities are: **Abs[x-s]**, **Cos[x,s]**, or **Dot[x,s]**.

We will see below that the ideal strategy is to compute the cross - correlation decision variable for each image (i.e. the dot product between each image data vector, say **x**, and an exact template of the signal, **s**, one is looking for), and pick the image which gives the larger cross - correlation.

# Probability Overview

For terminology, a fairly comprehensive outline, and overview, see notebook: ProbabilityOverview.nb in the syllabus web page, and for a general introduction, Griffiths and Yuille (2008).

For the section below, we'll use the properties of independence. Here is a quick overview of what we need today.

## Expectation & variance

Analogous to center of mass:

*Definition of expectation or average:*

$$\text{Average}[X] = \overline{X} = E[X] = \Sigma\, x[i]\, p[x[i]] \sim \sum_{i=1}^{N} x_i / N$$

$$\mu = E[X] = \int x\, p(x)\, \mathrm{dx}$$

Some rules:

E[X+Y]=E[X]+E[Y]

E[aX]=aE[X]

E[X+a]=a+E[X]

*Definition of variance:*

$$\sigma^2 = \text{Var}[X] = \text{E}[[\text{X-}\mu]\text{\^{}2}] = \sum_{j=1}^{N} ((p(x(j)))\,(x(j)-\mu))^2 = \sum_{j=1}^{N} \left(x_j-\mu\right)^2 p_j$$

$$\text{Var}[X] = \int (x-\mu)^2\, p(x)\, \mathrm{dx} \sim \sum_{i=1}^{N} (x_i-\mu)^2 \big/ N$$

*Standard deviation:*

$$\sigma = \sqrt{\text{Var}[X]}$$

Some rules:

$$\text{Var}[X] = E\left[X^2\right] - E[X]^2$$
$$\text{Var}[aX] = a^2\, \text{Var}[X]$$

## Statistics for independent random variables

Independence means that knowledge of one event doesn't change the probability of another event.

p(X)=p(X|Y)

p(X,Y)=p(X)p(Y) -- This is a key formula we will use below.

If p(X,Y)=p(X)p(Y), then

E[X Y] = E[X] E[Y] (i.e. X and Y are uncorrelated)

Var[X + Y] = Var[X] + Var[Y] (for uncorrelated random variables X and Y)

Var[c X] = $c^2$ Var[X], where c is a constant

# Ideal pattern detector for a signal which is exactly known ("SKE" ideal)

In this notebook we will study an ideal detector called the signal-known-exactly ideal (SKE). This detector has a built-in template that matches the signal that it is looking for. The signal is embedded in "white gaussian noise". "white" means the pixels are not correlated with each other--intuitively this means that you can't reliably predict what one pixel's value is from any of the others. (In general tho', lack of correlation doesn't necessarily imply independence.)

Assignment 1 simulates the behavior of this ideal. In the absence of any internal noise, this ideal detector behaves as one would expect a linear neuron to behave when a target signal pattern exactly matches its synaptic weight pattern. There are some neurons in the the primary cortex of the visual system called "simple cells". These cells can be modeled as ideal detectors for the patterns that match their receptive fields. In actual practice, neurons are not noise-free, and not perfectly linear.

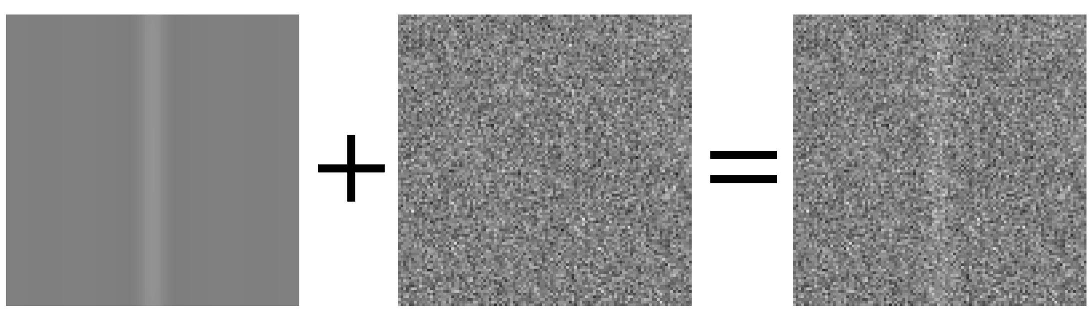
## Calculating the Pattern Ideal's d' based on signal-to-noise ratio

### ■ The signal + gaussian noise generative model

**x = s + n, where s is a vector of image intensities, e.g. corresponding to a face, snake, spot, ...or a gabor pattern**

**x = n, where n is a vector representing a sample of white gaussian noise**

Each element of **n** is assumed to have a mean of zero and standard deviation of $\sigma$. See the Exercise section below for *Mathematica* code of the generative process.



## Overview

We are going to do two things:

1. Show that a simple decision variable for detecting a known fixed pattern in white gaussian noise is the dot product, or cross-correlation, of the observation image **x** with the known signal image **s**.

$r = \mathbf{x} \bullet \mathbf{s}$ , or alternatively written as

$r = \sum_{i=1}^{N} x(i)\, s(i)$

2. Show that d' is given by:

$$d' = \frac{\sqrt{s \bullet s}}{\sigma}$$

**s** and **x** are a vectors, i.e. lists, of the image intensities, and $\sigma$ is the standard deviation of the added gaussian noise. Knowing the d' for the ideal will enable us to calculate the absolute efficiency for human visual detection.

## 1. Cross correlation produces an ideal decision variable: Proof

What is the optimal decision variable? Starting from the maximum a posteriori rule, we noted that basing decisions on the likelihood ratio is ideal, in the sense of minimizing the probability of error. So the likelihood ratio is a decision variable. But it isn't the only one, because any monotonic function is still optimal. So our goal is to pick a decision variable which is simple, intuitive, and easy to compute. But first, we need an expression for the likelihood ratio:

$$\frac{p\,(x \mid \text{signal plus noise})}{p\,(x \mid \text{noise only})} \tag{2}$$

where **x** is the vector representing the image measurements actually observed

**x = s + n, under signal plus gaussian noise condition**

**x = n, under gaussian noise only condition**

First let's consider just one pixel of intensity x. Under the signal plus noise condition, the values of x fluctuate (from one trial to the next) about the average signal intensity s with a Gaussian distribution (**gp[ ]**) with mean s and standard deviation $\sigma$.

So under the signal plus noise condition, the likelihood **p[x|s]** is the **gp[x-s; $\sigma$]**:

```
gp[x_,s_,σ_]:= (1/(σ*Sqrt[2 Pi])) Exp[-(x-s)^2/(2 σ^2)]
```

```
gp[x,s,σ]
```

$$\frac{e^{-\frac{(-s+x)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

Now consider the noise only condition. Again, consider just one pixel of intensity x. Under the noise only condition, the values of x fluctuate about the average intensity corresponding to the mean of the noise, which we assume is zero.

So under the  noise only condition, the likelihood p[x|n] is:

```
gp[x,0,σ]
```

$$\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

But we actually have a whole pattern of values of x, which make up an image vector **x**. So consider a pattern of image intensities represented now by a vector **x =** {x[1],x[2],...x[N]}. x[i] is the measured intensity at pixel i. s[i] would be the measured intensity at pixel i if the signal was displayed with no noise.

Let the mean values of each pixel under the signal plus noise condition be given by vector **s =** {s[1],s[2],...,s[N]}. The joint probability of an image observation **x**, under the signal hypothesis is:

```
Product[gp[x[i],s[i],σ],{i,1,N}]
```

$$\prod_{i=1}^{N}\frac{e^{-\frac{(-s[i]+x[i])^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

This is because we are assuming independence. In general, whether we can assume independence depends on the problem. In our case, the samples are independent by definition--as "experimenters" we generate the noise as independent samples.

Independence between pixels means we can multiply the individual probabilities to get the global joint image probability. (See above and **ProbabilityOverview.nb**)

The joint probability of an image observation **x**, under the noise only hypothesis is:

```
Product[gp[x[i],0,σ],{i,1,N}]
```

$$\prod_{i=1}^{N} \frac{e^{-\frac{x[i]^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

Now we have what we need for the likelihood ratio:

```
Product[gp[x[i], s[i], σ], {i, 1, N}] / Product[gp[x[i], 0, σ], {i, 1, N}]
```

$$\frac{\prod_{i=1}^{N} \frac{e^{-\frac{(-s[i]+x[i])^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}}{\prod_{i=1}^{N} \frac{e^{-\frac{x[i]^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}}$$

So at this point, we could just stop and write a program to use this product to make ideal decisions. E.g. if the product is bigger than 1, choose the signal hypothesis, and if less than 1 choose the noise hypothesis. But this is inefficient, and could be problematic because of the limitations in numerical precision (What if the computer rounds off just one of the factors in the above product to zero?).

But we can get a much simpler rule with a little more work.

This is because any monotonic function, **f()** of the likelihood ratio would give the same performance. A monotonic function simply means that whenever the **likelihood ratio is bigger than 1**, **f(likelihood ratio) is bigger than f(1).** So if we can find some monotonic function of the likelihood ratio that is simple, we'll have a simpler thing to calculate. The optimal decision rule will be to choose "signal" if **f(likelihood ratio)>f(1)**, and noise otherwise).

Let's try one--the natural logarithm will turn the product into a sum:

$$\mathrm{Log}\left[\frac{\prod_{i=1}^{N} \mathrm{gp}[x[i], s[i], \sigma]}{\prod_{i=1}^{N} \mathrm{gp}[x[i], 0, \sigma]}\right]$$

$$\mathrm{Log}\left[\frac{\prod_{i=1}^{N} \frac{e^{-\frac{(-s[i]+x[i])^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}}{\prod_{i=1}^{N} \frac{e^{-\frac{x[i]^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}}\right]$$

$$\text{Log}\left(\prod_{i=1}^{N} \frac{e^{-\frac{(x(i)-s(i))^2-x(i)^2}{2\sigma^2}}}{\sqrt{2\pi}\ \sigma}\right) \tag{3}$$

which is monotonic with:

$$\text{Log}\left[\prod_{i=1}^{N} e^{\frac{2\,x(i)\,s(i)}{2\sigma^2}}\right] \tag{4}$$

which simplifies to

$$\left(1/\sigma^2\right)\sum_{i=1}^{N} x(i)\,s(i) \tag{5}$$

But this is monotonic with:

$$r = \sum_{i=1}^{N} x(i)\,s(i) \tag{6}$$

In case that went by too fast, we can use *Mathematica*'s ability to handle symbolic expressions to see how to arrive at the same result. To be concrete, let N = 5.

```
Log[∏_{i=1}^{5} gp[x[i], s[i], σ] / ∏_{i=1}^{5} gp[x[i], 0, σ]]
```

```
Log[
 e^(x[1]^2/(2 σ^2) - (-s[1]+x[1])^2/(2 σ^2) + x[2]^2/(2 σ^2) - (-s[2]+x[2])^2/(2 σ^2) + x[3]^2/(2 σ^2) - (-s[3]+x[3])^2/(2 σ^2) + x[4]^2/(2 σ^2) - (-s[4]+x[4])^2/(2 σ^2) + x[5]^2/(2 σ^2) - (-s[5]+x[
 ]
```

Now use PowerExpand[] to simplify the above expression:

```
PowerExpand[%]
```

$$-\frac{1}{2\,\sigma^2}\left(s[1]^2+s[2]^2+s[3]^2+s[4]^2+s[5]^2-2\,s[1]\,x[1]-2\,s[2]\,x[2]-2\,s[3]\,x[3]-2\,s[4]\,x[4]-2\,s[5]\,x[5]\right)$$

Notice that the terms s[i] are fixed by definition (the "signal is known exactly"), so we can lump them together as a constant c.

$$\left(1/\sigma^2\right)\sum_{i=1}^{N} x(i)\,s(i)\ +\ c \tag{7}$$

But because we only care that the final function is monotonic with the likelihood ratio, we can drop the $\left(1/\sigma^2\right)$ and c terms:

```
Sum[x[i] s[i],{i,1,N}]
```

$$\sum_{i=1}^{N} s(i)\, x(i)$$

$$r = \sum_{i=1}^{N} x(i)\, s(i) \tag{8}$$

In other words, we've proven that the dot product, r, (or cross-correlation or matched filter) provides a decision variable which is optimal--in the sense that if we use the rule, the probability of error will be least. Now, let's calculate d'.

## 2. Derive formula for d'

By definition

```
d'=(μ2 - μ1)/σ
```

$$\frac{\mu 2 - \mu 1}{\sigma}$$

where u2 is the mean of the decision variable, r under the signal hypothesis (i.e. "switch set to send signal"), and u1 is the mean under the noise-only hypothesis (i.e. switch set to not send signal). (For our light discrimination example, $\mu 2$ = b, and $\mu 1$ =d)

To get d', we need formulas for the means and standard deviation for the decision variable, r under the two hypotheses, "signal plus noise" vs. "noise" only.

**First**, *suppose the switch is set for signal trials*. What is the average and standard deviation of r? I.e. $\mu 2$ and $\sigma$?

$$\begin{aligned}\mu 2 &= \text{Average}[r] = \text{Average}\left[\sum_{i=1}^{N} x(i)\, s(i)\right] = \\ &\sum_{i=1}^{N} \text{Average}[x(i)\, s(i)] = \sum \text{Average}[x(i)]\, s(i) = \sum s(i)\, s(i) = \sum s(i)^2\end{aligned} \tag{9}$$

$$\mu 2 = \sum_{i=1}^{N} s(i)^2 \tag{10}$$

(We've used the above rules: E[X+Y]=E[X]+E[Y], E[aX]=aE[X]. And because x(i) = s(i) + n(i), Average[x(i)]=s(i), using E[X+a]=a+E[X].)

And the variance is:

$$\mathrm{Var}\left(\sum_{i=1}^{N} x(i)\, s(i)\right) = \sum_{i=1}^{N} \mathrm{Var}[x(i)]\, s(i)^2 = \sigma^2 \sum_{i=1}^{N} s(i)^2 \tag{11}$$

We've used: Var[Y + Z] = Var[Y] + Var[Z], and Var[constant + n]=Var[n]. The s(i)'s are constant.

And, recall that Var[c Y] = c^2 Var[Y])

**Second**, suppose the switch is set for noise only trials. The average of the dot product is:

$$\mu 1 = \mathrm{Average}[r] = \mathrm{Average}\left[\sum_{i=1}^{N} x(i)\, s(i)\right] = \sum_{i=1}^{N} \mathrm{Average}[x(i)]\, s(i) = \sum_{i=1}^{N} 0\, s(i) = 0 \tag{12}$$

The variance is the same as for the signal case:

$$\mathrm{Var}\left(\sum_{i=1}^{N} x(i)\, s(i)\right) = \sum_{i=1}^{N} s(i)^2\, \mathrm{Var}[x(i)] = \sigma^2 \sum_{i=1}^{N} s(i)^2 \tag{13}$$

So d' is:

```
Sum[s[i]^2, {i, 1, N}]/Sqrt[(σ^2 Sum[s[i]^2,{i,1,N}])]
```

$$\frac{\sum_{i=1}^{N} s[i]^2}{\sqrt{\sigma^2 \sum_{i=1}^{N} s[i]^2}}$$

```
FullSimplify[Sum[s[i]^2, {i, 1, N}]/Sqrt[(σ^2 Sum[s[i]^2,{i,1,N}])],σ>0]
```

$$\frac{\sqrt{\sum_{i=1}^{N} s[i]^2}}{\sigma}$$

Or:

$$d' = \frac{\sqrt{\sum_{i=1}^{N} s(i)^2}}{\sigma} = \frac{\sqrt{s.s}}{\sigma} \tag{14}$$

## Calculating the Pattern Ideal's d' for a two-alternative forced-choice experiment from a z-score of the proportion correct.

Recall that for a 2AFC experiment, the observer gets two images to compare. One has the signal plus noise, and the other just noise. But the observer doesn't know which one is which. An ideal strategy is to compute the cross-correlation decision variable for each image, and pick the image which gives the larger cross-correlation. This strategy will result in a single number, the proportion correct, Pc. As we've seen before, d' for a 2AFC task can be calculated:

```
z[p_] := Sqrt[2] InverseErf[1 - 2 p];
```

where Z(*) is the z-score for $P_c$ , the proportion correct.

```
dprime[x_] := N[-Sqrt[2] z[x]]
```

Now try the psychophysics demo for pattern detection in noise

- **DEMO: GaborSKEDetection.nb**

# Next time

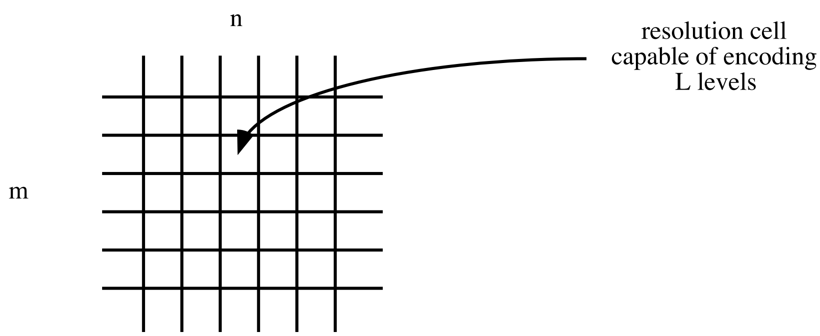
### High-level vision as Bayesian decision theory

- **Introduction to higher-level perceptual decisions as inference**
- **Bayesian decision theory**
- **Various types of inference Tasks: synthesis, inference (detection, classification, estimation), learning**

# Exercises

### Exercise: Calculate the information capacity of the eye

Consider an m x n pixel image patch. Is there a quantum limit to the number of light levels that can be represented in a resolution cell? (The size of a resolution cell is determined by the modulation transfer function of the optical device under consideration, which in this case would be the eye. We look later at how to estimate the spatial resolution of an imaging system).




Let SN be the maximum number of photons that land in a resolution cell. One can't discriminate this level from any other with an infinitely small degree of precision. Requiring a sensitivity of d', determines the next dimmest light level:

$$S_{N-1} = S_N - d'\sqrt{S_N}$$

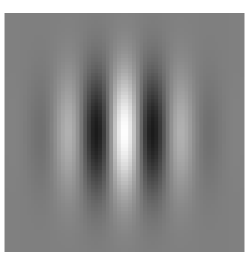
This effectively quantizes the dynamic range of a resolution cell. Write a small iterative program to count the number of levels down to S1 = zero. Say the number of levels is L, or $\mathrm{Log}_s L = l$ bits. Of course, one has to decide a priori what is a suitable discrimination level. But once done, the information capacity can be estimated by *lmn* bits.

## Generating gabor patch signals in additive noise

So what can you do with this particular ideal observer analysis? Take a look at:

Burgess, A. E., Wagner, R. F., Jennings, R. J., & Barlow, H. B. (1981). Efficiency of human visual signal discrimination. Science, 214, 93-94.

vs.



## The signal + gaussian noise generative model

**x = s + n, where s is a vector of image intensities corresponding to a gabor pattern**

**x = n, where n is white gaussian noise**

## Gabor patterns as signals

- **Basis set: Cartesian representation of Gabor functions:**

```
ndist=NormalDistribution[0,1];

cgabor[x_,y_, fx_, fy_,sx_,sy_] :=
Exp[-((x/sx)^2 +  (y/sy)^2)] Cos[2 Pi(fx x + fy y)];
```

- **Various frequencies , vertical orientations, and fixed width**

```
vtheta = Table[0, {i1,4}];
vf = {2,4};
hf = {0.0,0.0,0.0};
xwidth = {0.15,4};
ywidth = {4,4};
npoints = 128;
signalcontrast=0.15;
noisecontrast=0.2;
```

```
lr = -1; ur = 1; step = (ur - lr) / (npoints - 1);
signal =
  Table[signalcontrast cgabor[y, x, vf[[1]], hf[[1]], xwidth[[1]],
     ywidth[[1]]], {x, lr, ur, step}, {y, lr, ur, step}];
noise = noisecontrast Table[Random[ndist], {npoints}, {npoints}];
```

- **Signal, noise, signal + noise**

```
sig = ArrayPlot[signal, Mesh → False, Frame → False, PlotRange → {-1, 1},
   ColorFunction → "GrayTones"];
noi = ListDensityPlot[noise, Mesh → False, Frame → False,
   PlotRange → {-1, 1}, ColorFunction → "GrayTones"];
spn = ListDensityPlot[signal + noise, Mesh → False, Frame → False,
   PlotRange → {-1, 1}, ColorFunction → "GrayTones"];
```

```
GraphicsRow[{sig, noi, spn}]
```

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