

# Computational Vision

## U. Minn. Psy 5036

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#### Lecture 4: Ideal Observer Analysis

#### ■ Initialize standard library files:

```
<<Graphics`Graphics`  
<<Statistics`DiscreteDistributions`  
<<Statistics`ContinuousDistributions`  
Off[General::spell1];
```

We use `BarChart[]` in `Graphics.m` and `PoissonDistribution[]` in `DiscreteDistributions.m`.

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## Goals

One of the central take-home messages of this course is that perceptual decisions are limited by two primary sources:

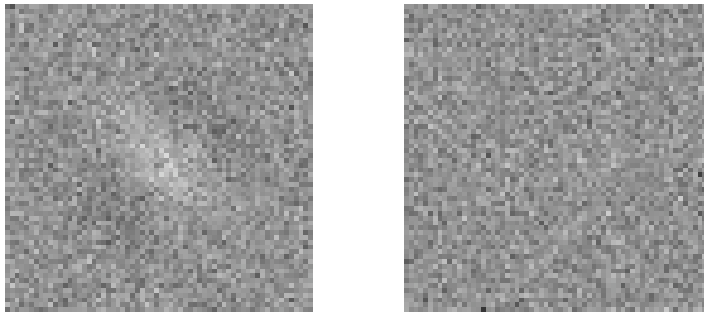
- 1) uncertainty in the stimulus information for a specific task
- 2) limitations of the human observer.

This is a critical distinction when drawing conclusions about the underlying neural mechanisms of the brain from behavioral/psychophysical data. The pattern of errors that a human observer shows when detecting patterns, discriminating vernier targets, finding targets in clutter, recognizing objects, reaching for objects, could in principle largely reflect the uncertainty in the task itself. If this is the case, then our simplest conclusion about the neural mechanism is that it behaves like an ideal observer, i.e. as a very efficient utilizer of the information available. On the one hand this is an economical description which is good; on the other, it limits our ability to draw conclusions about the neural mechanisms, which are often best revealed by sub-ideal behavior. Historically Hecht et al. and later Barlow and others used this comparison to good advantage. Hecht et al. argued that the variation in the proportion of hits was largely due to photon fluctuations, with only smaller contributions from limitations of the human observer, suggesting that the variability was due to a high efficiency of photon transduction.

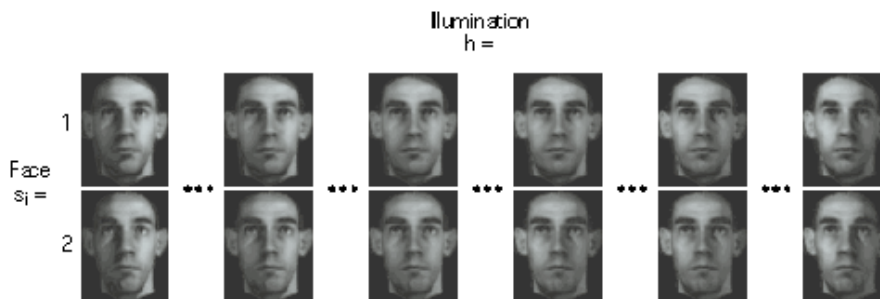
We'd like to further develop our tools of signal detection theory, and extend them to perceptual decisions more generally, so that we can quantitatively compare humans to ideal observers. We call this comparison *ideal observer analysis*.

Develop tools to analyze the information available in a task using ideal observer theory:

■ From dots to patterns: What does the eye see best?



■ From 2D image discrimination to object perception: How should we formulate object perception tasks?



Last time

■ External variability and the Ideal Observer for light discrimination

Introduced the idea of an **ideal observer** that models "external variability" and optimal decisions given that variability or uncertainty. We stated that the ideal observer should choose the hypothesis (i.e. switch setting) with the highest posterior probability, given the data (photon count). This decision process is called Bayesian inference. Bayesian theories of vision provide a quantitative model of the information available in a task. From there, we can test how well other observers (human, animal, or neuro) compare. The ideal can be used as a benchmark to measure the performance of humans, machines, and neurons for more complicated problems like pattern detection.

I asked you to accept (for the moment) that the Bayesian maximum a posteriori (MAP) decision rule is optimal in the sense of minimizing the total probability of error. Later we will prove this.

## Today

Our goals are:

- o Understand how to summarize ideal (and human performance) in the yes/no task in terms of hit and false alarm rates, and to relate these to  $d'$ . To do this, we will introduce the (standard) Gaussian approximation, and apply it to variability in light levels.
- o Introduce other tasks. In particular, the two-alternative forced-choice task
- o Understand how to quantitatively compare human and ideal performance.
- o Measure your own statistical efficiency in a 2AFC task

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## The standard Gaussian approximation

Most inference modeling is done using Gaussian models of variability. One reason is theoretical convenience. A deeper theoretical reason rests on the Central Limit Theorem, which says that a sum of random variables (from a non-Gaussian distribution) looks more and more Gaussian the more elements that are in the sum. Empirically, many experiments on human signal detection have been well-fit by assuming Gaussian distributions. However, as we will see later (when we measure statistics on natural images), the Gaussian assumption/approximation can fail miserably. It is always important to test this assumption. We'll first show that the Gaussian approximation provides a good approximation to the Poisson values.

Some terminology. We've adopted the convention of treating the high light as a "signal". Similarly, we can think of the chance low switch settings as "noise" to be taken into account. We will continue with this here, and begin to use the terms "signal" and "noise". But remember that this is just a convention--the problem is symmetric, and we could be talking about whether a measurement is from hypothesis A vs. hypothesis B.

## Gaussian approximation for signal and noise distributions

If  $a$  is large ( $a > 10$  or so), then the Poisson distribution (which is discrete) can be approximated very well by the Gaussian distribution (which is continuous):

$$p(X = x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}. \text{ The mean or expectation of } X \text{ is : } E(X) = \mu,$$

and the variance is :  $\text{var}(X) = \sigma^2$

This approximation is useful to estimate probability values for large  $a$ . If  $a$  is large enough, the probability of negative values (which is meaningless) is very small. For computational convenience and for later generality, we will usually use the Gaussian approximation.

Let's compare the forms of the Poisson and Gaussian distributions:

```

mean = 10;
poisson[x_,a_] := Exp[-a] a^x / Factorial[x]
p1 = Table[N[poisson[x,mean]], {x,0,50}];
g1=ListPlot[p1, PlotStyle->RGBColor[1, 0, 0],DisplayFunction->Identity];

```

```

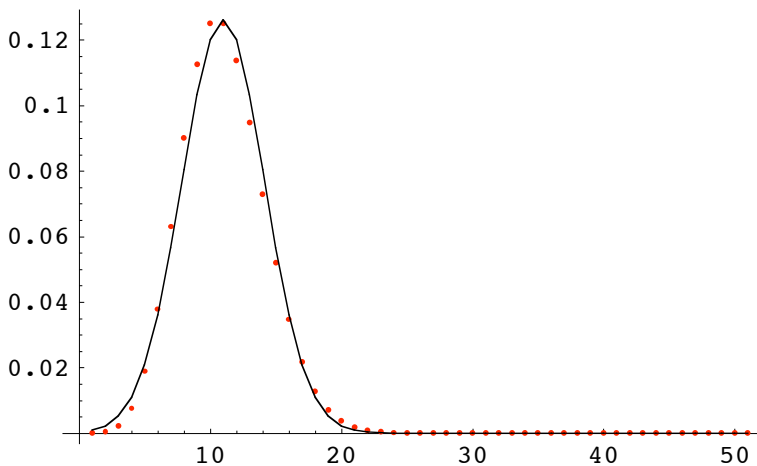
gauss[x_, mean_, std_] := Exp[-((x - mean) ^ 2) / (2 std ^ 2)] / (std Sqrt[2 Pi]);
p2 = Table[N[gauss[x, mean, Sqrt[mean]]], {x, 0, 50}];
g2 = ListPlot[p2, PlotJoined -> True, DisplayFunction -> Identity];

```

```

Show[{g1, g2}, DisplayFunction -> $DisplayFunction];

```



Try comparing the Poisson and Gaussian approximation when the mean is much smaller, like  $a=4$ .

### ■ Note: Discrete vs. continuous distributions

**Note.** We can interpret this approximation in two ways. We can discretize the continuous Gaussian function (as above) to give us a set of probabilities (over integers) that closely match those of the corresponding Poisson distribution, and make sure that the discrete sum is one (a fundamental requirement for a probability distribution). Alternatively, as the photon count gets high, we can treat light intensity as a continuous quantity (abandoning our quantized notion of light magnitude). In this latter case, we would treat the random variable  $X$  (light intensity) as being a continuous variable with a continuous probability distribution or "density". Then, because there is an infinite number of possible values over any finite range, the probability of  $X=x$ , for any particular value is actually zero! To fix this, we treat  $p(X)$  as a density (as in mass density), rather than a probability (as in mass). Then we can put a non-zero number on the probability of  $X$  taking on a value  $x$  in some small region,  $dx$  as:  $p(x < X < x+dx) \sim p(x)dx$ . More on this later.

## Summarizing performance

### ■ The classic *Standard Additive Gaussian* generative model for signal discrimination

Let's approximate our photon inspired model with a view towards generalization. We will express the generative model in a form that has become a core aspect of the classic signal detection theory. The so-called additive gaussian model is used for all kinds of detection tasks, for visual and auditory patterns, as well as non-perceptual decisions. We can model the shift of the peak of the distribution as an additive offset to the mean of a Gaussian. Then we have:

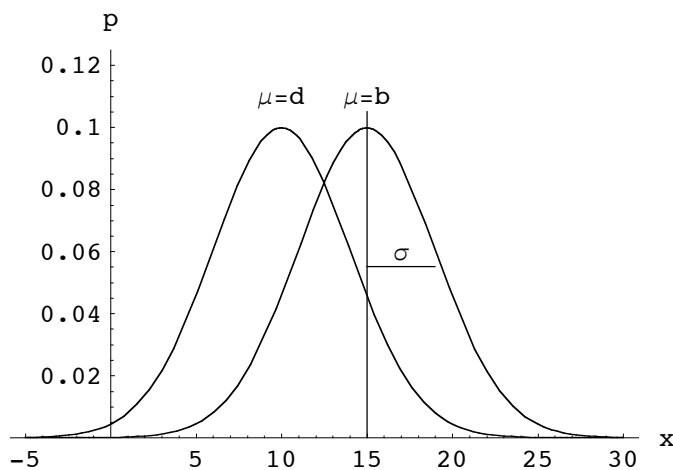
$$H = S_H: x = b + \text{noise};$$

$$H = S_L: x = d + \text{noise};$$

where noise is a Gaussian distributed random variable with mean,  $\mu = 0$ , and standard deviation  $\sigma$ . For the photon counting case, **b=highmean**, and **d=lowmean**. Note that the standard deviations of the high and low distributions would, for a Poisson distribution, be different (variance = mean for Poisson). We will assume that for a typical discrimination task, the distributions are quite close together, so the standard deviations are almost equal. The assumption of Gaussian distributions with equal variance is common, because it simplifies calculations, but more importantly because in many cases of sensory discrimination, the approximation is pretty good.

Here is a plot of the theoretically predicted histograms for a signal (high) mean of 15, a noise (low) mean of 10, and a standard deviation of 4 for each:

```
b = 15; d = 10;
Plot[{gauss[x, b, 4], gauss[x, d, 4]}, {x, -5, 30}, AxesLabel -> {"x", "p"},
PlotRange -> {0, 0.125},
Epilog -> {Text["μ=b", {b, 0.11}], Text["μ=d", {d, 0.11}],
Text["σ", {b + 2, 0.06}], Line[{{b, 0.055}, {b + 4, 0.055}}],
Line[{{b, 0}, {b, 0.105}}]};
```



### ■ The signal-to-noise ratio: $d'$ , a summary statistic for ideal performance

By using the Gaussian distributions (with equal variances), we can characterize the ideal's signal-to-noise ratio with one number, the "signal-to-noise ratio," defined as  $d'$ :

$$d' = \frac{b-d}{\sigma}$$

where  $b$  and  $d$  are the high and low means, respectively. This makes intuitive sense. Discrimination should get easier as the difference between the means increases (the "signal" is the difference) or as the spread given by the standard deviation of the additive noise decreases—hence *signal-to-noise ratio*. Note that  $d'$  can also decrease by increasing  $\sigma$ .

But what does this mean in terms of performance?

### ■ The decision criterion, costs, benefits, hits, false alarms

How does the ideal observer make a decision as to whether the low or high light was flashed? Earlier we derived a criterion starting from the assumption that we wanted to maximize the posterior probability ( $p(H|x)$ ) over  $H$ :

$$1) \quad \operatorname{argmax}_H p(H|x)$$

From here, we showed that if the prior probabilities over the hypotheses were the same ( $p(H=S_H) = p(H=S_L)$ ), this was equivalent to maximizing the likelihood:

$$2) \quad \operatorname{argmax}_H p(x|H)$$

Because we are considering only two hypotheses, we could reformulate the decision strategy to testing whether the ratio

$\frac{p(x|S_H)}{p(x|S_L)} > 1$ ? This in turn, is equivalent to testing:  $\log\left[\frac{p(x|S_H)}{p(x|S_L)}\right] > 0$ ? And from there, we showed that for photon counting, the decision could be simply related to testing whether the photon count was bigger than a particular criterion (call it  $X_T$ ) determined by two light level means ( $b$  and  $d$ ).

Graphically, the ideal makes its decision by saying "high" if the measurement  $x$  is right of the cross-over point on the above plot.

I stated that this minimizes the probability of error, but how is error related to the decision criterion and the distributions?

Let's consider the more general case, where the criterion isn't necessarily at the point we calculated (which is where the two likelihood functions cross over).

### ■ Hit, false alarm (positive), miss, and correct rejection rates.

It is easy to see how to measure the signal-to-noise ratio for light discrimination for the ideal observer—we just collect histograms under the two conditions ( $H=S_L$  and  $H=S_H$ ), approximate them by Gaussian distributions, and assuming the standard deviations are close, use these Gaussian fits to estimate  $d'$ .

But how can we measure the signal-to-noise ratio, or  $d'$  of a human observer? Human decisions are based on some hidden, and probably quite complex neural mechanism in the brain. We'd have to have access to the neural response that behaves like the ideal's decision variable, but is consistent with human performance, which is usually sub-ideal. This is an interesting scientific problem, but let's see if we can put a number on human  $d'$ , without "going inside the box".

To answer this question, let's take a look at an alternative way of estimating  $d'$  for the ideal observer. The ideal observer (or receiver) for light intensity discrimination has two ways of being right and two ways of being wrong:

### ■ Two ways of being right and two ways of being wrong

Being correct:

it can score a

**hit** (e.g. says "high" when the switch was set to high)

or a

**correct rejection** (e.g. says "low" when the switch was set to low)

Being incorrect:

it can suffer a

**false alarm** (also called "false positive") (e.g. says "high" when the switch was set to low)

or a

**miss** (or "false negative") (e.g. says "low" when the switch was set to high)

(Statisticians talk about a similar distinction in terms of Type II and Type I errors).

### ■ Rates

Average performance in a yes/no task is completely characterized by calculating the proportions of two of the four. For example,

$$\text{hit rate} = \frac{\# \text{ times observer says high when switch was set to high}}{\# \text{ times switch was set to high}}$$

~ p(decide high | switch set on high)

$$\text{false alarm (positive) rate} = \frac{\# \text{ times observer says high when switch was set to low}}{\# \text{ times switch was set to low}}$$

~ p(decide high | switch is set to low)

We only need measures of these two because the correction rejection and miss rates are not independent of the hit and false alarm rates. The corresponding correct rejection and miss rates are:

$$p(\text{correct rejection}) = 1 - p(\text{false alarm}), \text{ and } p(\text{miss}) = 1 - p(\text{hit}), \text{ respectively.}$$

Sometimes, we talk about the average error rate. Since there are two ways of being wrong: Deciding "high" when  $H = S_L$ , and deciding "low", when  $H = S_H$ . The total error rate is the (weighted) average of the miss and false alarm rates. The error rate is determined by the mean values for the high and low settings. As  $b$  increases, the separation between the probability distributions increases, and the overlap decreases, so the error rate decreases. So intuitively, there should be some relationship between  $d'$  and error and/or success rates.

**How should one compute weighted average for error rate?**

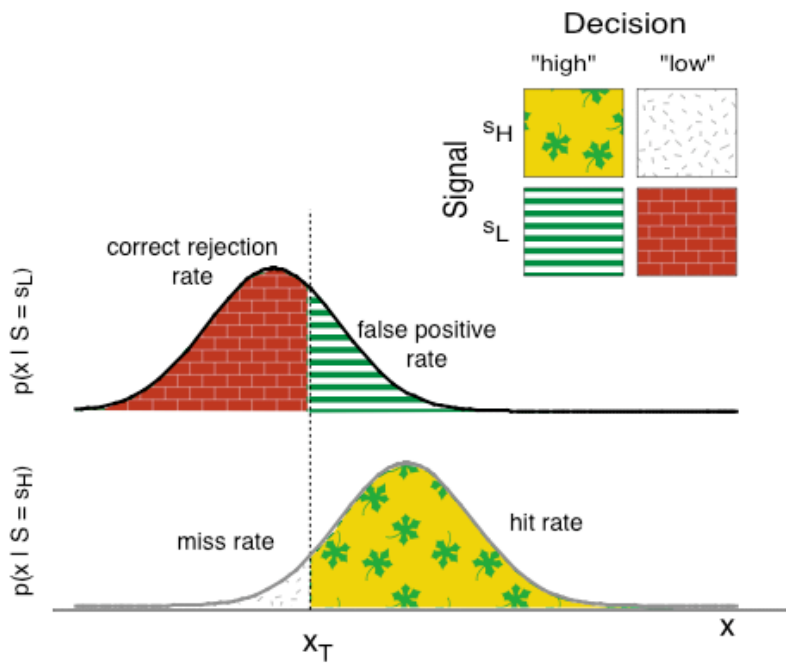
■ **Graphical view of the hit, false alarm rate, ...**

For a probability density function (often referred to as "PDF"), say  $p(x)$ , the probability of a measurement  $X$  falling within a certain range is given by the area under the density over that range:

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} p(x) dx$$

$$P(X > X_T) = \int_{X_T}^{\infty} p(x) dx$$

The criterion, and thus the hit and false alarm rates could be determined by the relative costs or benefits (loss or gain) one assigns to a particular choice of hit and false alarm rates. Suppose the criterion is  $X_T$ , in general not at the cross-over point of the likelihoods. Then the hit rate (PH) is determined by the area under the (signal or high) curve to the right of  $X_T$ . The false alarm rate (PFA) is given by the area under the ("noise" or low) curve to the right of  $X_T$ .





### ■ Criterion shifts affect hit and false alarm rates, but don't affect basic sensitivity

A light is flashed, the photon counter indicates  $x$  photons received. The decision rule is:

if  $x > XT$  say "high was flashed"

if  $x \leq XT$  say "low was flashed"

In general, where the criterion gets placed depends on the decision goal. One could have other goals (than minimizing error) that would determine where to put the criterion level. Put yourself in the place of an ideal (not a MAP observer) with the following constraints:

o If you were hit every time you said high, you would never say high--you would never get any hits. This in effect pushes the criterion far to the right.

o If you liked chocolates as much as I do, and received a sweet every time you said high, you would always say high, even if it was not presented-- you would have many false alarms. This pushes the criterion far to the left.

So the goal doesn't have to be determined by maximizing the proportion of correct responses, it can be determined by other criteria, which in turn modify the decision rule. These other factors can be incorporated into a pay-off matrix (see Green and Swets, 1974).

Later we will look at formalizing and generalizing the notions of costs and benefits as statistical decision theory. For example, the cost to an error in an estimate of illumination can be low as compared to a cost in the error of face identification (see figure at the beginning of the lecture).

### ■ Summary of modified log likelihood rule:

Basically, one can derive a simple modification of the log likelihood rule:

Rather than testing:

$$\log\left[\frac{p(x|S_H)}{p(x|S_L)}\right] > 0?,$$

decide using:

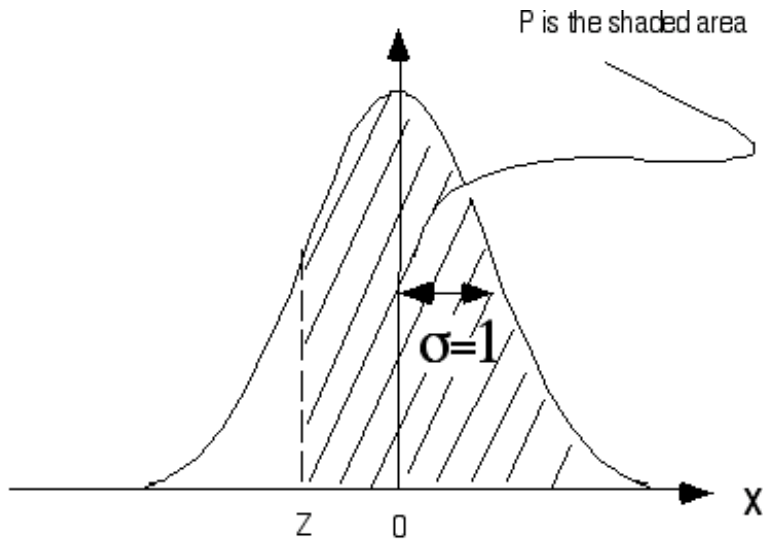
$$\log\left[\frac{p(x|S_H)}{p(x|S_L)}\right] > k?,$$

where  $k$  is a function of the costs and benefits.

### ■ Relationship between signal-to-noise ratio ( $d'$ ) and hit and false alarm rates.

We are now ready to produce an important result. We noted that the signal-to-noise ratio  $d'$  can be estimated from the means and standard deviation. But if we don't have access to those numbers (as happens in psychophysics), it turns out that with a bit of mathematics, one can show that  $d'$  can be obtained from the hit and false alarm rates using the following formula:

$$d' = z(\text{PFA}) - z(\text{PH})$$



$P = \int_z^{\infty} \text{gauss}[x, 0, 1] dx$ . And  $z(p)$  is the inverse. There is no simple formula for  $z$ , but there are good closed-form approximations. *Mathematica* doesn't give the direct formula for the z-score, but it does supply the inverse of the erf[] function, which comes from the engineering (rather than statistics) tradition:

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

As you will use in Assignment #1, the z-score function is related to the InverseErf function by:

```
z[p_] := Sqrt[2] InverseErf[1 - 2 p];
```

## Humans vs. ideals: Modeling internal variability of the human observer

### The basic idea: human observers are sub-ideal, but still "ideal-like"

The idea is to model the human signal discrimination as being "ideal-like" in assuming that human decisions respect an implicit generative model:

$$H = S_H: x = b^h + \text{noise}^h;$$

$$H = S_L: x = d^h + \text{noise}^h;$$

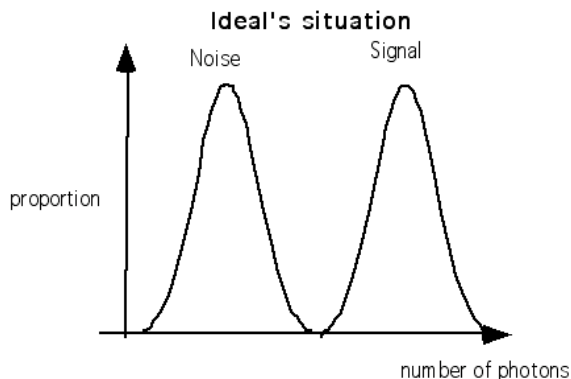
where  $b^h$ ,  $d^h$ , and noise<sup>h</sup> are the equivalent states of the world that would give rise to the human's  $d'$  (as measured from hit and false alarm rates). It is as if the human visual system does Bayesian inference (is ideal), but has the wrong generative model--i.e. a different state of the world. Note that there is indeterminacy in these "implicit" variables,  $b^h$ ,  $d^h$ , and  $\sigma^h$  (the standard deviation of noise<sup>h</sup>). Because sensitivity, the  $d'$  for human, is determined by hit and false alarm rates, or equivalently by:

$$d' \text{ for human} = \frac{b^h - d^h}{\sigma^h}, \text{ there is an infinite family of combinations of } b^h, d^h, \text{ and } \sigma^h \text{ which give the same } d'.$$

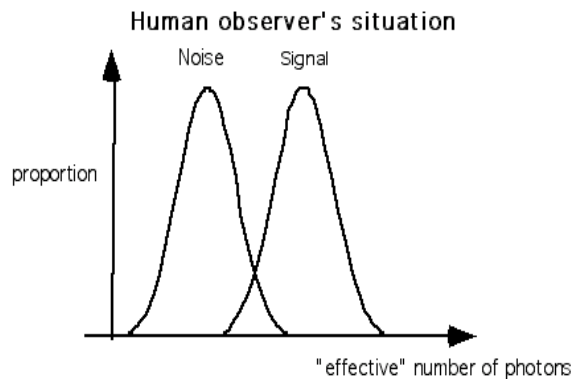
How good is the model? One way of testing it is to plot hit and false alarm rates for human decisions and compare them to this "sub-ideal" that has additive gaussian noise with equal variances. Surprisingly often, the model fits are quite good. But first, let's see how we can make an absolute comparison of performance.

### Comparing ideal and human performance

The physics of the experiment determines the mean levels  $d$ ,  $b$  and the standard deviation. We have seen that the ideal's performance is characterized by one number called the sensitivity  $d'$ . Now that we understand the limitations on the performance of an ideal observer, let us try to understand how to compare human performance to the ideal. Even if the ideal is making near perfect discriminations, the human observer may not be doing so well because of other sources of uncertainty. For example, the ideal may be contending with the following situation:



We can't "see" or directly measure the distributions that the human observer is using to make the decision, but we can suppose that it is based on distributions that are in effect much closer together:



Although we can not measure the human's  $d'$  by measuring the separation between these two distributions and their standard deviations (they are not directly measurable), we've seen that the human (or ideal) observer's performance can be determined by the hit and false alarm rate:

$$d' \text{ for human} = z(\text{false alarm rate for human}) - z(\text{hit rate for human})$$

### ■ Statistical efficiency

Given the means to compute  $d'$  for the ideal and for the human observer in the same task, we can compare them. Usually we calculate the ideal's  $d'$  from the signal-to-noise ratio, and the human's from the hit and false alarm rate in a yes/no task or from the proportion correct in a 2AFC task (as in the experiment below). (The ideal's  $d'$  could be calculated from its hit and false alarm rates but this usually isn't as convenient--but it might serve as a good way to double-check that you are doing the right calculations.)

With these two  $d'$ 's in hand we can compare the performances of the two observers. One way is in terms of *statistical efficiency*. Efficiency is defined as the number of samples (e.g. photons) required by the ideal divided by the number of samples required by the human, when they are performing equivalently (e.g. same hit and false alarm rates). It can be shown that:

$$\text{Statistical efficiency} = (d' \text{ for human} / d' \text{ for ideal})^2$$

### ■ Historical note -- *Quantum Efficiency accounting for the missing information*

In 1962, Horace Barlow reported results on the measurements quantum efficiency for light discrimination (rather than detection) under low light (scotopic) conditions similar to those of Hecht et al., and came up with a figure for QE of about 10%. That is, the human observer behaved like an ideal observer who was only receiving one out of every ten photons. Where was the photon loss coming from? Like we saw for Hecht et al., Barlow traced the losses to reflection, scatter and absorption by the optic media, and losses due to photons falling in the spaces between the rods, and an imperfect isomerization efficiency. Recall that a figure of 10% is close to what one would predict from Hecht et al.'s experiment.

Barlow later went one step beyond Hecht et al.. He concluded (Barlow, 1977) that there was still a residual inefficiency even after taking into account all the above causes, which he calculated as accounting for only 80% of the photon loss. He was left with about 50% of human discrimination efficiency due to limitations in the brain's ability to "count" point events. That is, for example, if 100 photons are incident on at the cornea of the eye, about 20 of these are reliably transduced and this information is sent to the brain. But he argued, the brain deals with this average of 20 photons with 50% efficiency--that is, the ideal's "brain" could discriminate just as well with only an average of 10 photons. Barlow made this latter conclusion by a clever argument involving a psychophysical experiment in which he had observers discriminate differences in dot density (rather than photon density) on a CRT screen. The idea was that although the presence of a photon at the retina does not necessarily make it to the brain, a dot will.

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## Psychophysical tasks & techniques

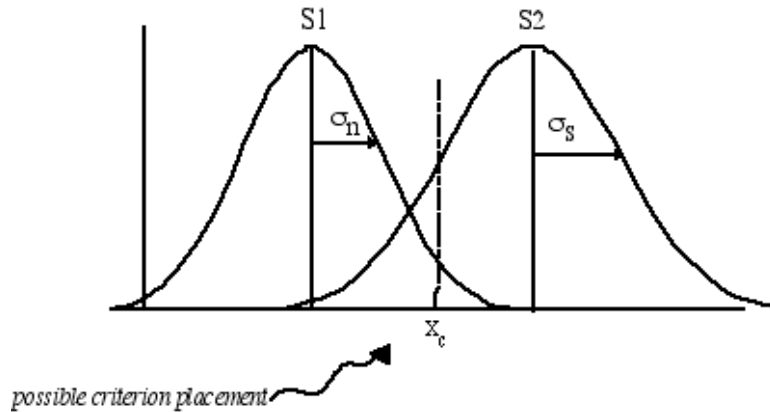
### The Receiver Operating Characteristic (ROC)

Although we can't directly measure the internal distributions of a human observer's decision variable, we've seen that we can measure hit and false alarm rates, and thus  $d'$ . But one can do more, and actually test to see if an observer's decisions are consistent with Gaussian distributions with equal variance. If the criterion is varied, we can obtain a set of  $n$  data points:

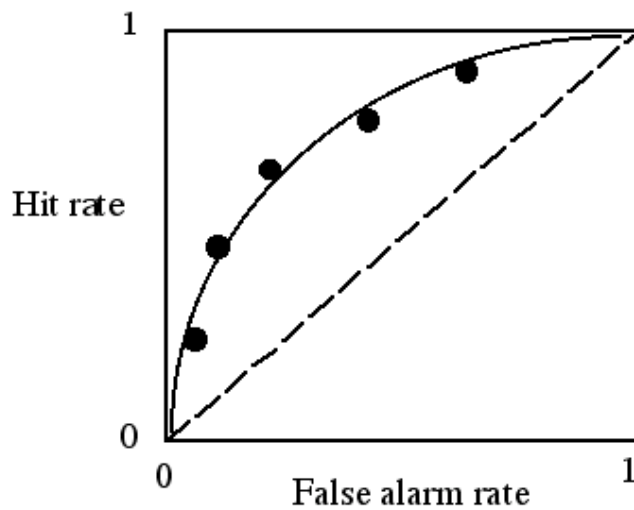
{(hit rate 1, false alarm rate 1), (hit rate 2, false alarm rate 2), ..., (hit rate  $n$ , false alarm rate  $n$ )}

all from one experimental condition (i.e. from one signal-to-noise ratio, call it  $d'_{ideal}$ ). This is because as the hit rate varies, so does the false alarm rate. One could compute the  $d'$  for each pair and they should all be equal for the ideal observer. Of course, we would have to make a large number of measurements for each one--but on average, they should all be equal.

To get meaningful and equal  $d$ 's for each pair of hit and false alarm rates assumes that the underlying relative separation of the signal and noise distributions remain unchanged and that the distributions are Gaussian, with equal standard deviation. We might know this is true (or true to a good approximation) for the ideal, but we have no guarantee for the human observer. Is there a way to check? Suppose the signal and noise distributions look like:



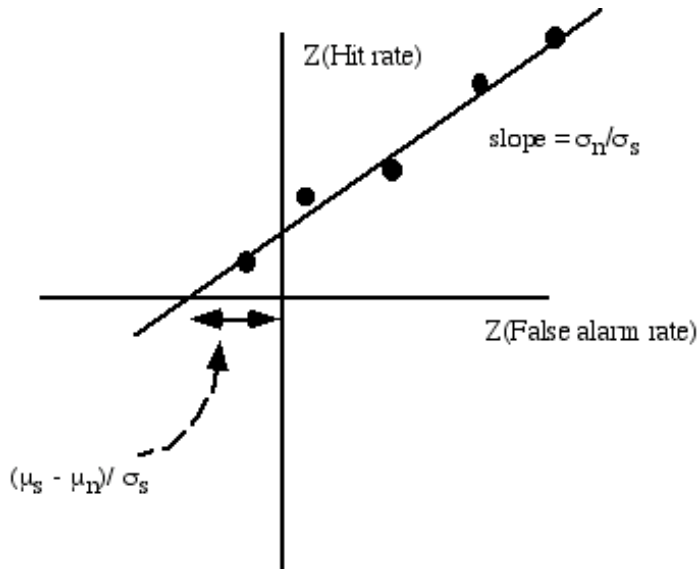
If we plot the hit rate vs. false alarm rate data on a graph, usually we get something that looks like:



One can show that the area under this ROC curve is equal to the proportion correct in a two-alternative forced-choice experiment (Green and Swets). (Sometimes, sensitivity is operationally defined as this area. This provides a single summary number, even if the standard definition of  $d'$  is inappropriate, for example because the variances are not equal.)

We return to our basic question: is there a way to spot whether our gaussian equal-variance assumptions are correct for human observers?

If we take the same data and plot it in terms of Z-scores we get something like:



In fact, if the underlying distributions are Gaussian, the data should lie on a straight-line. If they both have equal variance, the slope of the line should be equal to one. This is because:

$$Z(\text{hit rate}) = \frac{X_c - \mu_s}{\sigma_s}$$

$$Z(\text{false alarm rate}) = \frac{X_c - \mu_n}{\sigma_n}$$

And if we solve for the criterion  $X_c$ , we obtain:

$$Z(\text{hit rate}) = \frac{\sigma_n}{\sigma_s} Z(\text{false alarm rate}) - \frac{\mu_s - \mu_n}{\sigma_s}$$

(I've switched notation here, where  $b = \mu_s$ , and  $d = \mu_n$ ). The main point of this plot is to see if the data tend to fall on a straight line with slope of one. If a straight line, this would support the Gaussian assumption. A slope = 1 supports the assumption of equal variance Gaussian distributions.

In practice, there are several ways of obtaining an ROC curve in human psychophysical experiments. One can vary the criterion that an observer adopts by varying the proportion of times the signal is presented. As observers get used to the signal being presented, for example, 80% of the time, they become biased to assume the signal is present. One needs to block trials in groups of, say 400 trials per block. Where the signal and noise priors are fixed for a given block.

One can also use a *rating scale* method in which the observer is asked to say how confident she/he was (e.g. 5 definitely, 4 quite probable, 3 don't know for sure, 2, unlikely, 1 definitely not). Then we can bin the proportion of "5's" when the signal vs. noise was present to calculate hit and false alarm rates for that rating, do the same for the "4's", and so forth. The assumption is that an observer can maintain not just one stable criterion, but four---the observer has in effect divided up the decision variable domain into 5 regions. An advantage of the rating scale method is efficiency---relatively few trials are required to get an ROC curve. Further, in some experiments, ratings seem psychologically natural to make. But if there is any "noise" in the decision criterion itself, e.g. due to memory drift, or whatever, this will act to decrease the estimate of  $d'$  in both yes/no and rating methods.

Usually rather than manipulating the criterion, we would rather do the experiment in such a way that it does not change. Is there a way to reduce the problem of a fluctuating criterion?

## The 2AFC (two-alternative forced-choice) method

### ■ Relating performance (proportion correct) to signal-to-noise ratio, $d'$ .

In psychophysics, the most common way to minimize the problem of a varying criterion is to use a two-alternative forced-choice procedure (2AFC). In a 2AFC task the observer is presented on each trial a pair of stimuli. One stimulus has the signal (e.g. high flash), and the other the noise (e.g. low flash). The order, however, is randomized. So if they are presented temporally, the signal or the noise might come first, but the observer doesn't know which from trial to trial. In the spatial version, the signal could be on the left of the computer screen with the noise on the right, or vice versa. One can show that for 2AFC:

$$d' = \sqrt{2} z \text{ (proportion correct)} \quad (1)$$

### Exercise: Prove $d' = \sqrt{2} z \text{ (proportion correct)}$

If you want to prove this for yourself, here are a couple of hints--actually, a lot of hints. Let us imagine we are giving the light discrimination task to the ideal observer. We have two possibilities for signal presentation: Either the signal is on the left and the noise on the right, or the signal is on the right and the noise on the left. There are two ways of being right. The observer could say "on the left" when the signal is on the left, or "on the right" when the signal is on the right. For example, for the light detection experiment, a reasonable guess is that all the ideal observer would have to do is to count the number of photons on the left side of the screen and count the number on the right too. If the number on the left is bigger than the number on the right, the observer should say that the signal was on the left. Thus, a 2AFC decision variable would be the difference between the left and right decision variables, where each of these is what we calculated for the yes/no experiment.

$$r = r_L - r_R \quad (2)$$

For example as you will see in Assignment 1,  $r_L$  and  $r_R$  for the SKE observer would be the dot products of the signal pattern template with observation image vectors on the left and right sides.

So, the probability of being correct is:

$$pc = p(r > 0 | \text{signal on left}) p(\text{signal on left}) + p(r < 0 | \text{signal on right}) p(\text{signal on right})$$

What is the probability distribution of  $r$ ? Well, from our probability rules,

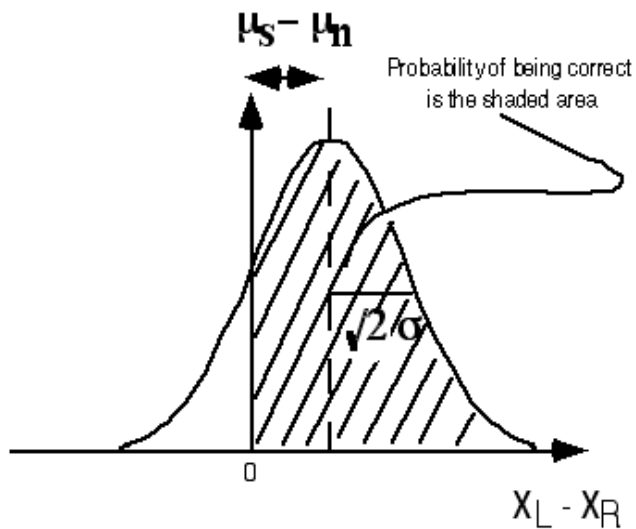
$$\text{average}(r) = \mu_2 - \mu_1 = \mu_s - \mu_n$$

$$\text{var}(r) = \text{var}(r_L) + \text{var}(r_R) = \sqrt{2} \sigma_{r_L}$$

(Because the mean of the sum of two independent random variables is the sum of their means and that the variance of the sum is the sum of the variances.)

If the signal is equally likely to appear on the left or the right, the probability of being correct is the area under the curve to the right of zero of the distribution of  $r$ :





■ **Calculating the Pattern Ideal's  $d'$  for a two-alternative forced-choice experiment from a z-score of the proportion correct.** (see Homework Assignment #1)

For our 2AFC experiment, the observer gets two images to compare. One has the signal plus noise, and the other just noise. But the observer doesn't know which one is which. An ideal strategy is to compute the cross-correlation decision variable for each image, and pick the image which gives the larger cross-correlation. This strategy will result in a single measurable number, the proportion correct,  $P_c$ .

$d'$  for a 2AFC task is given by the formula:

$$d' = -\sqrt{2}Z(P_c)$$

$$P = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-x^2/2} dx$$

■ **Z-score.** You used the inverse of a standard mathematical function called **Erf[]** to get  $Z$  from a measured  $P$ .

```
z[p_] := Sqrt[2] InverseErf[1 - 2 p];
```

where  $Z(*)$  is the z-score for  $P_c$ , the proportion correct.

```
dprime[x_] := N[-Sqrt[2] z[x]]
```

### ■ *Adaptive procedures for finding thresholds using 2AFC or yes/no.*

There have been a number of advances in the art of efficiently finding values of a signal which produce a certain percent correct in a psychophysical task such as 2AFC. For more on this, see the QUEST procedure of Watson and Pelli (1983) and the analyses of Treutwein (1993). See: <http://vision.arc.nasa.gov/mathematica/psychophysica/>

## Your statistical efficiency: 2AFC task

### ■ Set up the mini-experiment

Let's develop the dot density experiment you piloted last time. We'll make two improvements. First we'll turn it into a two-alternative forced-choice experiment. You'll experience how this makes task subjectively easier, and reduces the problem of criterion fluctuation. Second, we will calculate your and the ideal's proportion correct, turn these into d's, and from there calculate your statistical efficiency. In 1997, Barlow reported an efficiency near 50%. How good are you?

As above, let's define a Poisson distribution with a mean of **mean**, with a function to draw a sample from this distribution.

For most psychophysical experiments it is a good idea to give the observer some practice trials. For these, set **highmean = 300** for the "high" setting, and **lowmean = 200** for the "low" setting. Set **numtrials=10**; This will give you easy trials to get the hang of it

For the actual measurements, you want to make the task hard enough so that mistakes are made (why?). So make the task harder by setting **highmean = 220** for the "high" setting, and **lowmean = 200** for the "low" setting. Set **numtrials=100**;

```
numberofphotons [mean_] := Random[PoissonDistribution [mean] ] ;
highmean = 220; lowmean = 200;
data = {{"Was I Correct?", "Was Ideal Correct?"}};

numtrials = 10;
```

### ■ Execute a trial

Now, randomly turn the switch to "high" or "low", draw a sample, and then input your response (1 for "high" and 0 for "low"). Execute the next cell 100 times.

```

Table[whichflash = Random[Integer, {0, 1}];
  If[whichflash == 1, leftnumsample = numberofphotons [highmean],
    leftnumsample = numberofphotons [lowmean]];
  If[whichflash == 0, rightnumsample = numberofphotons [highmean],
    rightnumsample = numberofphotons [lowmean]];

leftsample = Table[{Random[], Random[]}, {leftnumsample}];
rightsample = Table[{Random[], Random[]}, {rightnumsample}];

leftsampleleg = Graphics[{PointSize[0.01], Point /@ leftsample},
  AspectRatio → 1, Frame → False, FrameTicks → None,
  Background → GrayLevel[0.0], PlotRange → {{-.2, 1.2}, {-.2, 1.2}}];
rightsampleleg = Graphics[{PointSize[0.01], Point /@ rightsample},
  AspectRatio → 1, Frame → False, FrameTicks → None,
  Background → GrayLevel[0.0], PlotRange → {{-.2, 1.2}, {-.2, 1.2}}];

Show[GraphicsArray[{leftsampleleg, rightsampleleg}, GraphicsSpacing → .3],
  Frame → False];

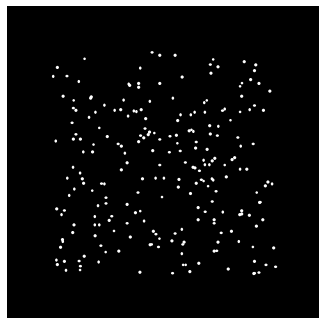
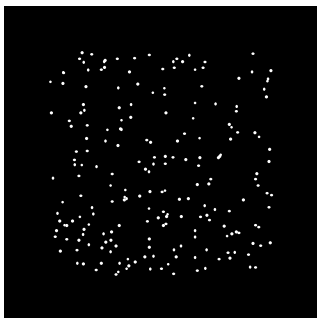
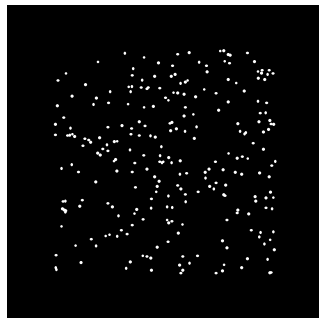
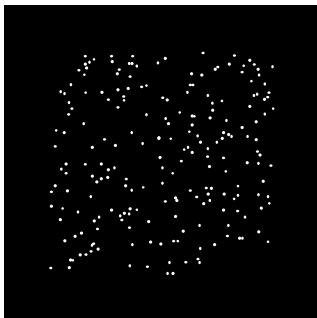
myanswer = Input["SignalonLeft?"];
If[myanswer == whichflash, WasICorrect = 1, WasICorrect = 0];

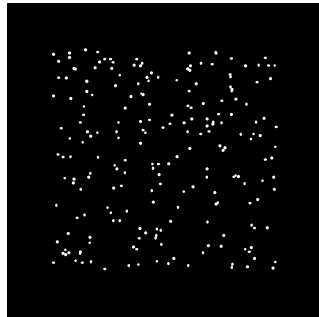
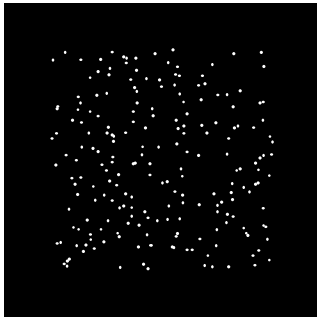
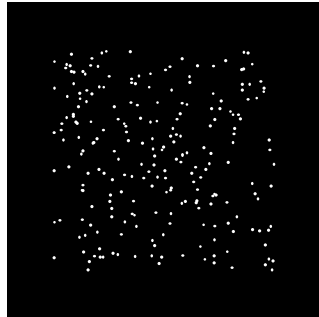
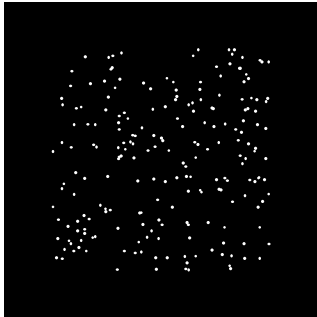
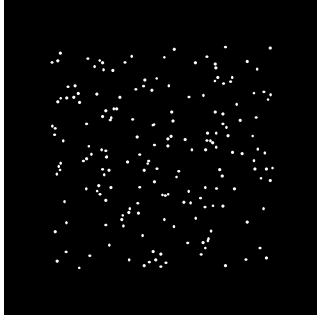
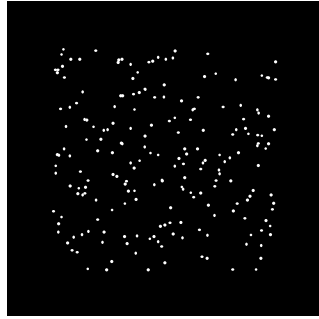
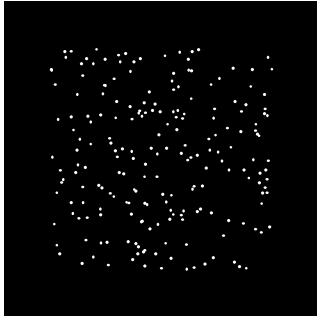
idealanswer = If[(leftnumsample > rightnumsample), 1, 0];

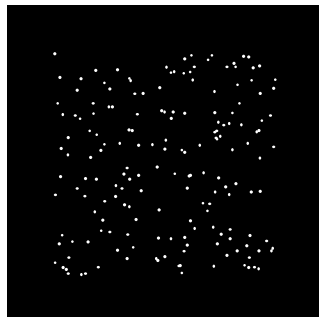
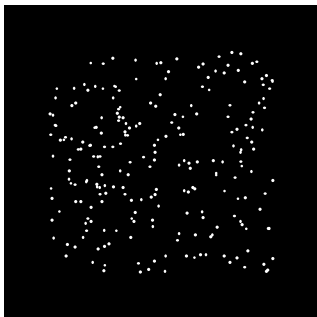
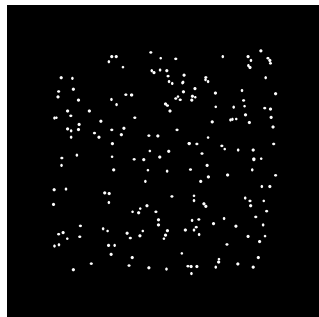
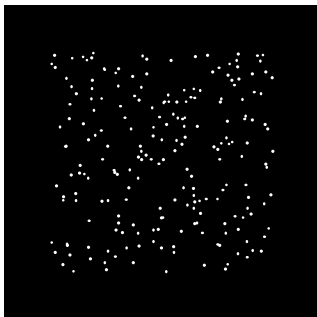
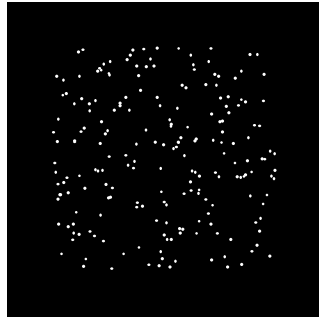
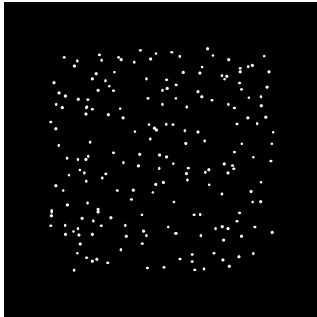
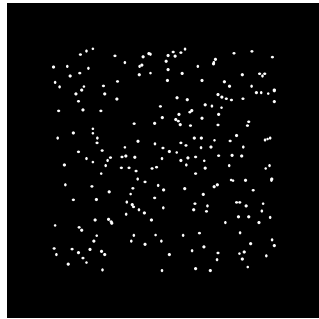
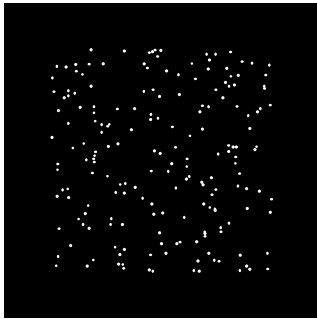
If[idealanswer == whichflash, WasIdealCorrect = 1, WasIdealCorrect = 0];

data = Append[data, {WasICorrect, WasIdealCorrect}], {numtrials}
];

```







### ■ Display the data

```
data // MatrixForm
```

$$\begin{pmatrix} \text{Was I Correct?} & \text{Was Ideal Correct?} \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

### ■ Analyze the data

Let's drop the table heading stored in row 1, and then transpose the matrix so that the columns become the rows:

```
data2 = Transpose[Drop[data, 1]]
```

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Let's use a combination of Map[] and Count[] (used earlier to make histograms) to count up all occurrences of an event type. So the total for myhits is:

```
myproportioncorrect =
  N[Map[Count[data2[[1]], #] &, {1}] / Dimensions[data2][[2]][[1]];
idealproportioncorrect =
  N[Map[Count[data2[[2]], #] &, {1}] / Dimensions[data2][[2]][[1]];

mydprime = dprime[myproportioncorrect];
idealdprime = dprime[idealproportioncorrect];
mystatisticalefficiency = Round[100 * (mydprime / idealdprime)];
```

```
Print[{"my proportion correct", "ideal's proportion correct",
      "my dprime", "ideal's dprime", "my efficiency (%)"},
      {myproportioncorrect, idealproportioncorrect, mydprime,
       idealdprime, mystatisticalefficiency}];
```

```
( my proportion correct  ideal's proportion correct  my dprime  ideal's dprime  my efficiency (%) )
(           0.7                0.7                0.741614   0.741614                100 )
```

To get a reliable estimate, you need at least 100 or more trials.

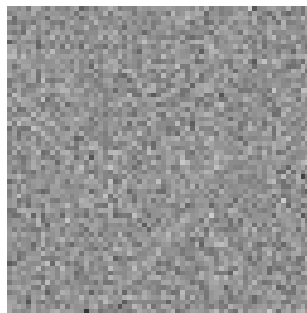
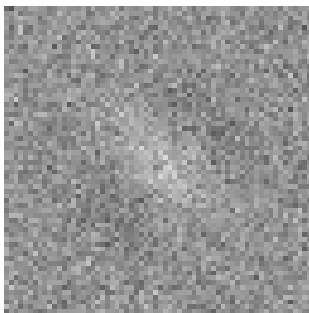
What can you do if the human psychophysical observer is making lots of mistakes and the ideal doesn't make any? The above method is called the "method of constant stimuli", because (although the stimuli really aren't constant), the conditions (highmean and lowmean) are. Adaptive or tracking methods are more efficient. The idea is to have a computer program automatically hunt for that threshold (e.g. highmean is adjusted) so that the observer is getting a prescribed proportion correct (e.g. 75%). See Watson and Pelli (1983). For *Mathematica* tools, see: <http://vision.arc.nasa.gov/mathematica/psychophysica/>.

---

## Next time

### Probability overview

### From pixels to patterns: What does the eye see best?



- Computing the ideal observer for patterns

- Comparing psychophysical performance for pattern detection with properties of visual neurons in the brain

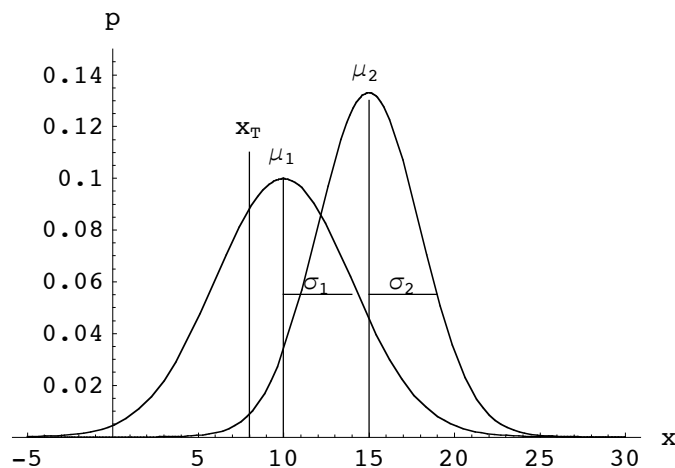
- Preview of assignment #1.

## Exercises

### Figure code

```
gauss[x_,  $\mu$ _,  $\sigma$ _] := PDF[NormalDistribution[ $\mu$ ,  $\sigma$ ], x]
```

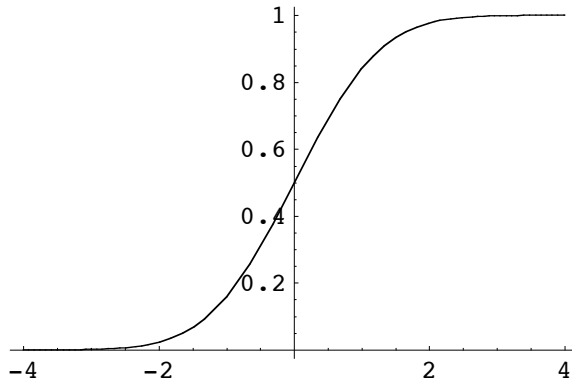
```
b = 15; d = 10; sb = 3; sd = 4;
p1 = Plot[{gauss[x, b, sb], gauss[x, d, sd]}, {x, -5, 30},
  Background -> GrayLevel[1], AxesLabel -> {"x", "p"}, PlotRange -> {0, 0.15},
  Prolog -> {Text[" $\mu_1$ ", {d, 0.11}], Text[" $\mu_2$ ", {b, 0.143}, {0.1, 0.1}],
  Text[" $\sigma_1$ ", {d + 2, 0.06}], Text[" $\sigma_2$ ", {b + 2, 0.06}],
  Line[{{b, 0.055}, {b + 4, 0.055}}, Line[{{d, 0.055}, {d + 4, 0.055}}],
  Line[{{d, 0}, {d, 0.1}}, Line[{{d - 2, 0}, {d - 2, 0.1}}],
  Text[" $x_T$ ", {d - 2, 0.12}], Line[{{b, 0}, {b, 0.13}}]
}];
```



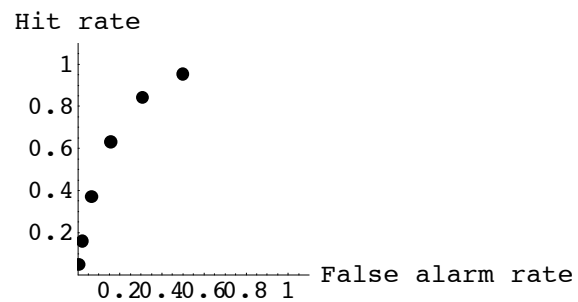


```
z[p_] := Sqrt[2] InverseErf[1 - 2 p];
```

```
cumulgauss[x_, μ_, σ_] := CDF[NormalDistribution[μ, σ], x];
Plot[cumulgauss[x, 0, 1], {x, -4, 4}];
```



```
hitrate = N[Table[1 - cumulgauss[xt, d, sd], {xt, 10, 20, 2}]];
falsealarmrate = N[Table[1 - cumulgauss[xt, b, sb], {xt, 10, 20, 2}]];
ROC = Table[{hitrate[[i]], falsealarmrate[[i]]}, {i, Length[hitrate]}];
ListPlot[ROC, PlotRange -> {{0, 1.1}, {0, 1.1}},
  AxesLabel -> {"False alarm rate", "Hit rate"}, AspectRatio -> 1,
  Prolog -> AbsolutePointSize[5]];
```



```

dg = ListPlot[z[ROC], AxesLabel → {"Z[False alarm rate]", "Z[Hit rate]"},
  AspectRatio → 1, DisplayFunction → Identity,
  PlotRange → {{-1, 4}, {-3, 5}},
  Prolog → {AbsolutePointSize[5], Text["x intercept =  $\frac{\mu_2 - \mu_1}{\sigma_2}$ ", {2, 4.0}],
    Text["Slope =  $\frac{\sigma_1}{\sigma_2}$ ", {3.2, 1}]}];

```

```
z[ROC]
```

```

( 0.  -1.66667 )
( 0.5 -1. )
( 1.  -0.333333 )
( 1.5  0.333333 )
( 2.  1. )
( 2.5  1.66667 )

```

```
Fit[z[ROC], {1, x}, x]
```

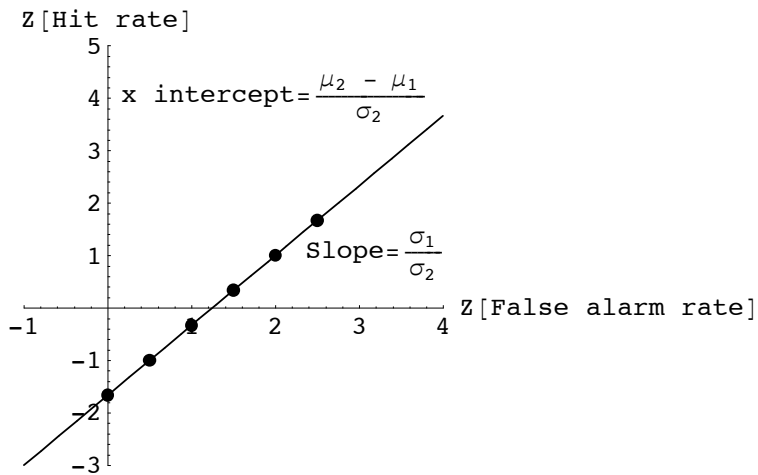
```
1.33333 x - 1.66667
```

```

fg = Plot[Evaluate[Fit[z[ROC], {1, x}, x], {x, -1, 4},
  PlotRange → {{-1, 4}, {-3, 5}}, DisplayFunction → Identity];

```

```
Show[dg, fg, DisplayFunction -> $DisplayFunction];
```



## Probability and statistical sampling

Goal: Learn how to use *Mathematica* to generate probability distributions, cumulative distributions and do statistical sampling.

### Distributions and densities

The probability of  $x$  photons being detected by an ideal detector is given by the Poisson distribution:

```
poisson[x_, a_] := Exp[-a] a^x / Factorial[x];
```

where  $a$  is the mean.

**Exercise:** What is the probability of detecting 12 photons if the mean is 10? Generate more than 20 values and then use the function `Apply[Plus, t1]`, to demonstrate that the sum over all values is 1.

Whenever it is convenient in this course, we will make use of predefined functions in *Mathematica*.. There are two packages **DiscreteDistributions.m**, and **ContinuousDistributions.m** which contain the definitions of continuous distributions, cumulative distributions, and provide the means to draw samples. At the top of this notebook, to load the necessary functions, we used the commands :

```
<<Statistics`DiscreteDistributions`
```

```
<<Statistics`ContinuousDistributions`
```

Let's define a Poisson distribution with a mean of 20:

```
pdist = PoissonDistribution[20];
```

The probability distribution function (PDF is given by:

```
PDF[pdist, x]
```

$$\frac{20^x}{e^{20} x!}$$

You can obtain the mean, variance and standard deviation of the distribution we've defined. Try it:

```
Mean[pdist]  
Variance[pdist]  
StandardDeviation[pdist]
```

```
20
```

```
20
```

```
 $2\sqrt{5}$ 
```

The output shows *Mathematica's* definition of the function. The "If"'s test to make sure that x is not negative and is an integer. The rest of the definition should look familiar. In this lecture, we make use of the fact that the continuous normal density can provide a good approximation to the Poisson distribution when the mean is large enough and if we set the standard deviation to **Sqrt[20]**:

```
ndist = NormalDistribution[20, 4.47214];
```

```
N[PDF[pdist,28]]
N[PDF[ndist,28]]
```

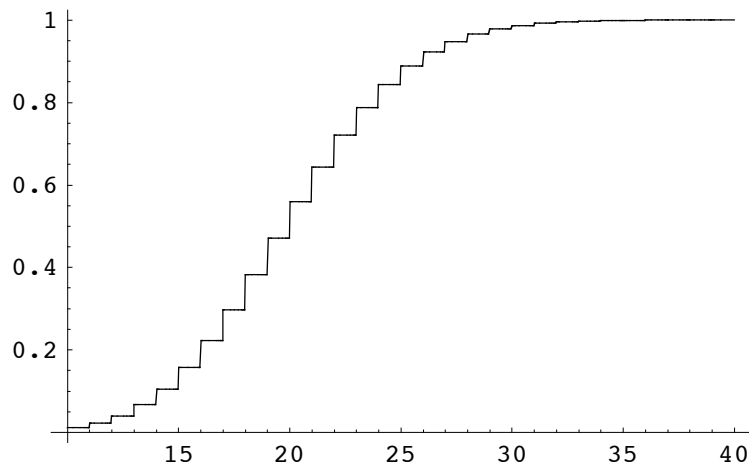
```
0.0181472
```

```
0.0180105
```

## The cumulative distribution

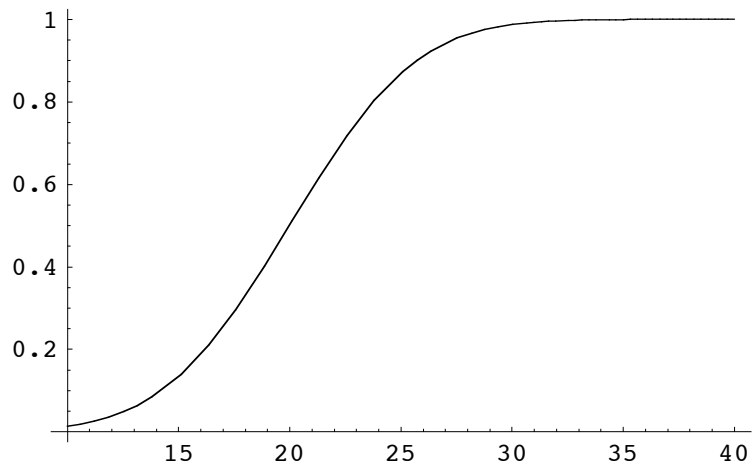
The cumulative distribution gives the probability that the detector signals  $x < k$  photons. It is obtained by adding up the probabilities for all values less than  $k$ . For the cumulative density function, we integrate over all values less than  $k$ . The packages we've loaded (**DiscreteDistributions.m**, and **ContinuousDistributions.m**) already have built in the definitions for the cumulative distributions. Here is the cumulative distribution for the discrete Poisson distribution with a mean of 20:

```
Plot[CDF[pdist,x],{x,10,40}];
```



What is the probability of detecting 50 or less photons when the mean is 20? It is virtually certain-- as you can see from the graph, the probability is almost 1. Here is the plot of the continuous normal distribution with a mean of 20, and a standard deviation of **Sqrt[20]**:

```
Plot[CDF[ndist,x],{x,10,40}];
```



---

## References

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